Quantum Physics of Light-Matter Interactions

Summer Semester 2019, FAU, Erlangen
Class structure

• Lectures: 12 each of two hours (classes cancelled on bridge days: May 31 and June 21)
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• Lecturer: Claudiu Genes – MPL Group *Cooperative Quantum Phenomena* (see website where lecture notes are available)

• Exercises: 12 each for one hour (Michael Reitz)
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- Exam: 90 minutes written/10 minutes oral examination
Fundamentals of light-matter interactions

- The two-level system approximation
- Light-matter interaction – dipole and rotating wave approximations
- The master equation for spontaneous emission
- Solving the master equation – optical Bloch equations
- Fundamentals of cavity quantum electrodynamics – the Jaynes Cummings Hamiltonian and quantum Langevin equations

The laser

- Laser threshold
- Laser photon statistics

Effects of light onto motion

- The full quantum optical force
- Doppler cooling, gradient cooling
- Ion traps, ion cooling, ion logic
- Cavity optomechanics

Superradiance/subradiance

- Collective decay of coupled quantum emitters
- Subradiantly enhanced energy transfer

Molecules: vibronic coupling

- The Holstein Hamiltonian
- Emission and absorption spectra, energy transfer
Consider a quantum emitter: atom, molecule, quantum dot etc
How does it interact with electromagnetic waves?
Simplified picture – two level system
Light absorption

$|e\rangle \quad \bullet$

$|g\rangle$
Story in short

Light emission

\[ |e\rangle \rightarrow |g\rangle \]
There is also a quantum vacuum of electromagnetic modes (quantization in a big box)

\[ |e\rangle \quad \bullet \]

\[ |g\rangle \quad \_ - \_

Story in short
Effect – spontaneous emission – gives rise to finite lifetime (non-zero linewidth) for electronic transitions

- Master equation description of the dynamics
Effect – spontaneous emission – gives rise to finite lifetime (non-zero linewidth) for electronic transitions

- Master equation description of the dynamics
- Optical Bloch equations
Optical cavity

- Multiple resonances
Optical cavity

- Multiple resonances
- Lorentzian profile – enhanced density of optical modes around resonances

Cavity mode linewidth

Quasi-mode

Frequency
The Jaynes-Cummings Hamiltonian

\[ H_{JC} = \hbar g \left[ a^\dagger \sigma + a \sigma^\dagger \right] \]

\[ g = \frac{1}{\hbar} \mathcal{E}_0 \varepsilon_{eg} = \left[ \frac{\omega}{\hbar \varepsilon_0 l S} \right]^{1/2} \varepsilon_{eg} \]
Enhancement of photon-emitter scattering – cavity quantum electrodynamics

- Jaynes-Cummings Hamiltonian
- Quantum Langevin equation for the cavity field
- Input-output relations
Goals

Fundamentals of light-matter interactions

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Source: youtube: How a laser works
Can you build a (theoretical) laser?
Enhancement of photon-emitter scattering – optical cavities

Pump leads to population inversion

Non-thermal light?
Model 1

- Pump
- Fast relaxation
- Lasing transition
- Laser threshold
Model 2

Lasing transition

\( |2\rangle \quad \rightarrow \quad |4\rangle \)

\( |1\rangle \quad \rightarrow \quad |3\rangle \)

\( \rho_F(t) \)

\( \mathcal{P}(\Delta t) \)

\( \rho_F(t + \Delta t) \)

Laser photon statistics
Outline of lectures

Fundamentals of light-matter interactions

- The two-level system approximation
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Molecules: vibronic coupling

- The Holstein Hamiltonian
- Emission and absorption spectra, energy transfer
Light can affect motion!

Light carries momentum: \( \hbar k = \hbar \omega / c \)

Recoil momentum – recoil energy:

\[
\hbar \omega_{\text{rec}} = \frac{(\Delta p)^2}{2m} = \frac{(\hbar k)^2}{2m}
\]
Laser cooling

Possible processes:

- Stimulated absorption

- Stimulated absorption

- Spontaneous emission

\[ \hbar k = \hbar \omega / c \]
Light can be used to control motion! (for example cooling)

Counterpropagating fields

\[ F_{\text{standing}}(z) = (F_0 + \beta v) \hat{z} + (-F_0 + \beta v) \hat{z} = 2\beta v \hat{z} \]

\[ \beta = \hbar k_L^2 \frac{2\gamma \Omega^2 \Delta}{[\gamma^2 + \Delta^2]^2} \]

Effective Langevin equation

\[ \frac{d\hat{p}}{dt} = -f(\hat{p}) + \hat{p}_{in} \]
Polarization gradient cooling force

More complicated cooling schemes (Sysiphus cooling)
Light can control motion even at the macroscale!

Macroscopic mechanical resonator

Can it be brought close to the quantum ground state of vibrations?
Free space

- Single pass
- Weak interaction

Recoil momentum – recoil energy

\[ \hbar \omega_{rec} = \frac{(\Delta p)^2}{2m} = \frac{(\hbar k)^2}{2m} \]

- Single pass
- Weak interaction

<table>
<thead>
<tr>
<th>State</th>
<th>Species</th>
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<tr>
<td>(</td>
<td>e\rangle )</td>
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<td>(</td>
<td>g\rangle )</td>
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Cavity optomechanics

**Free space**
- Single photon
- Single pass
- Weak interaction

**Optical cavity**
- Single photon
- Multiple bounces (100000 or more)
- Stronger interaction

Standing wave modes (like on a guitar string)

- |e⟩ for Atoms
- |g⟩ for Fluorescent molecules, Ions, Quantum dots
Original motivation: improving precision in the detection of gravitation waves
**Goal:** use light to control motion of massive mechanical resonators at the quantum level.

### Free space

- Radiation pressure force
- Tiny effect

### Cavity enhancement

- Dynamical system – back action of motion onto the cavity field
- Time-delayed equations
- Both amplification and cooling of mechanical motion achievable
- Nonlinear dynamics – chaos, self-oscillations

### Towards nano-optomechanics

- LIGO, VIRGO
- Detection of gravitational waves

### Basic science

- Testing the classical-quantum boundary at the large mass scale

### Technology

- Ultra-sensitive displacement detection
- Inertial sensors

A macroscopic resonator in a thermal environment

\[ \dot{q} = \omega_m p, \]
\[ \dot{p} = -\gamma_m p - \omega_m q + \zeta(t) \]
A macroscopic resonator in a thermal environment

\[ \dot{q} = \omega_m p, \]

\[ \dot{p} = -\gamma_m p - \omega_m q + \zeta(t) \]

Stochastic noise term

Induced damping

\( \zeta(t) \)
A macroscopic resonator in a thermal environment

\[
\begin{align*}
\dot{q} &= \omega_m p, \\
\dot{p} &= -\gamma_m p - \omega_m q + \zeta(t)
\end{align*}
\]

Stochastic noise term

Induced damping

\[\zeta(t)\]

Noise correlations

\[
\langle \zeta(t)\zeta(t') \rangle = \frac{\gamma_m}{\omega_m} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \left[ \coth \frac{\hbar \omega}{2k_B T} + 1 \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} S_{th}(\omega)
\]
A macroscopic resonator in a thermal environment

\[
\dot{q} = \omega_m p, \quad \dot{p} = -\gamma_m p - \omega_m q + \zeta(t)
\]

**Stochastic noise term**

**Induced damping**

**Noise correlations**

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\langle \zeta(t) \zeta(t') \rangle = \frac{\gamma_m}{\omega_m} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \omega \left[ \coth \frac{\hbar \omega}{2k_B T} + 1 \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} S_{th}(\omega)
\]

\[
S_{th}(\omega_m) = 2\gamma_m (\bar{n} + 1)
\]

\[
S_{th}(-\omega_m) = 2\gamma_m \bar{n}
\]

\[
\bar{n} \approx \frac{k_B T}{\hbar \omega_m}
\]

**Thermal noise spectrum**

\[
\gamma_m = \frac{S_{th}(+\omega_m) - S_{th}(-\omega_m)}{2}
\]

\[
\bar{n} = \frac{S_{th}(-\omega_m)}{2\gamma_m}
\]
A macroscopic resonator in a thermal environment

\[ \dot{q} = \omega_m p, \quad \dot{p} = -\gamma_m p - \omega_m q + \zeta(t) \]

**Stochastic noise term**

**Induced damping**

\[ \zeta(t) \]

**Noise correlations**

\[
\langle \zeta(t) \zeta(t') \rangle = \frac{\gamma_m}{\omega_m} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \omega \left[ \coth \frac{\hbar \omega}{2k_B T} + 1 \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} S_{th}(\omega)
\]

\[
S_{th}(\omega_m) = 2\gamma_m (\bar{n} + 1)
\]

\[
S_{th}(-\omega_m) = 2\gamma_m \bar{n}
\]

\[
\bar{n} \approx \frac{k_B T}{\hbar \omega_m}
\]

**Equipartition!**

\[
\langle q^2(t) \rangle = \bar{n} + \frac{1}{2}
\]

\[
\langle p^2(t) \rangle = \bar{n} + \frac{1}{2}
\]

**Thermal noise spectrum**

\[
\gamma_m = \frac{S_{th}(+\omega_m) - S_{th}(-\omega_m)}{2}
\]

\[
\bar{n} = \frac{S_{th}(-\omega_m)}{2\gamma_m}
\]

Cooling scheme analogous to sideband cooling of trapped ions!
Laser cooling

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Molecules: vibronic coupling

- The Holstein Hamiltonian
- Emission and absorption spectra, energy transfer
How do atoms decay when they see each other?
Answer: faster (superradiance) or slower (subradiance) than independent atoms
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Atoms: complicated fine and hyperfine structure

Molecules: fundamentally more complex

Not always a good enough model
A simple model - diatomic molecules
A simple model for electronic to vibrations coupling in diatomic molecules

\[ H = \left( \omega_e + \frac{p^2}{2\mu} + \frac{1}{2} \mu \nu^2 (R - R_e)^2 \right) \sigma^\dagger \sigma + \left( \frac{p^2}{2\mu} + \frac{1}{2} \mu \nu^2 (R - R_g)^2 \right) \sigma \sigma^\dagger \]
Molecular systems – coupling to vibrations

H atom H atom

Internuclear Distance

Energy

Dissociation Level

Excited State (a' Δg)

Vibrational Levels

Ground State (X^3Σ_g)

MakeAGIF.com
Molecular systems – coupling to vibrations

\[ H = \nu b^\dagger b + (\omega_e + \lambda^2 \nu)\sigma^\dagger \sigma - \lambda \nu (b^\dagger + b)\sigma^\dagger \sigma \]

Holstein Hamiltonian
Förster resonance energy transfer

Fig. 1 The acceptor and donor fluorophores must be closer than ~10 nm for the energy transfer to occur (adopted from P. I.H. Bastiaens and R. Pepperkok, TIBS 25 (2000) 631).

An Introduction to Fluorescence Resonance Energy Transfer (FRET) Technology and its Application in Bioscience, Paul Held, Ph.D, Applications Department, BioTek Instruments (2005)
The two level system
Interaction with a classical light field
Simplifying assumptions:

- nucleus is heavy and therefore fixed in the origin
- non-relativistic description (Schrödinger equation suffices)
- spinless electron and proton

Radially symmetric potential (Coulomb) – exactly solvable model

\[
i\hbar \frac{\partial}{\partial t} \psi = H \psi
\]

\[
H = \frac{\hat{P}^2}{2\mu} + V(|\vec{r}|)
\]
In Dirac (ket) notation Hamiltonian is diagonalized

\[
H = \sum E_{nlm} |nlm\rangle \langle nlm|
\]

\[
E_n = - \left[ \frac{\mu e^4}{32\pi^2 \varepsilon_0^2 \hbar^2} \right] \frac{1}{n^2}
\]

\[
(1 \leq n < \infty, 0 \leq \ell \leq n - 1, -\ell \leq m \leq \ell)
\]
In the position representation

\[ i\hbar \frac{\partial}{\partial t} \psi (r, t) = \left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \psi (r, t) \]

Solutions

\[ \psi_{nm} (r, \theta, \phi) = \langle r | nm \rangle = R_{nm} (r) Y_{l,m} (\theta, \phi). \]
In the position representation

\[ i\hbar \frac{\partial}{\partial t} \psi(r, t) = \left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \psi(r, t) \]

Solutions

\[ \psi_{n\ell m}(r, \theta, \phi) = \langle r | n\ell m \rangle = R_{n\ell}(r) Y_{\ell m}(\theta, \phi). \]
Interaction with classical light

Propagating field

\[ E(x, y, z, t) = E_L f(x, y) \cos(k_L z - \omega_L t) \hat{\epsilon}_z = \frac{iE_L}{2} \left( e^{ik_L z} e^{-i\omega_L t} - e^{-ik_L z} e^{i\omega_L t} \right) \hat{\epsilon}_z \]
Interacting with classical light

**Propagating field**

\[
E(x, y, z, t) = E_L f(x, y) \cos(k_L z - \omega_L t) \hat{e}_z = \frac{iE_L}{2}(e^{ik_L z}e^{-i\omega_L t} - e^{-ik_L z}e^{i\omega_L t})\hat{e}_z
\]

**Interaction Hamiltonian (in the dipole approximation)**

**Classical minimal coupling Hamiltonian**

\[
H(\mathbf{r}, t) = \frac{1}{2m}(p + eA(\mathbf{r}, t))^2 - e\Phi(\mathbf{r}, t) + V(\mathbf{r})
\]

- **Vector potential**
- **Scalar potential**
Interaction with classical light

**Propagating field**

\[ E(x, y, z, t) = E_L f(x, y) \cos(k_L z - \omega_L t) \hat{e}_z = \frac{iE_L}{2} (e^{ik_L z} e^{-i\omega_L t} - e^{-ik_L z} e^{i\omega_L t}) \hat{e}_z \]

**Interaction Hamiltonian (in the dipole approximation)**

**Classical minimal coupling Hamiltonian**

\[ H(r, t) = \frac{1}{2m} (p + eA(r, t))^2 - e\Phi(r, t) + V(r) \]

**Length gauge transformation + quantization**

**Dipole Hamiltonian**

\[ H = \frac{\hat{p}^2}{2\mu} + V(|\hat{r}|) - \hat{d} \cdot \hat{E}(t) \]
A bit more on the Hamiltonian in the dipole approximation

\[
H(r, t) = \frac{1}{2m} (p + eA(r, t))^2 - e\Phi(r, t) + V(r)
\]

\[
A(r, t) = A(t)
\]

\[
H(r, t) = \frac{p^2}{2m} + \frac{e}{m}A(t) \cdot p + \frac{e^2}{2m}A^2(t) + V(r)
\]

Gauge transformation with \( \chi(r, t) = -A(t) \cdot r \)

\[
\begin{align*}
\Phi'(r, t) &= -\frac{\partial \chi(r, t)}{\partial t} = -\frac{\partial A(t)}{\partial t} \cdot r = r \cdot E(t) \\
A'(r, t) &= A(t) + \nabla \chi(r, t) = A(t) - \nabla (A(t) \cdot r) = 0
\end{align*}
\]

\[
H' = \frac{p^2}{2m} + V(r) - d \cdot E(t)
\]
In more detail

**Dipole Hamiltonian**

\[
H = \frac{\hat{p}^2}{2\mu} + V(|\hat{r}|) - \hat{d} \cdot \mathbf{E}(t)
\]

\[
-\hat{d} \cdot \mathbf{E}(t) = -\frac{iE_L}{2}(e^{-i\omega_L t} - e^{i\omega_L t}) [-e\hat{r}] \cdot \hat{e}_z
\]
In more detail

**Dipole Hamiltonian**

\[ H = \frac{\hat{p}^2}{2\mu} + V(|\hat{r}|) - \hat{d} \cdot \mathbf{E}(t) \]

\[-\hat{d} \cdot \mathbf{E}(t) = -\frac{iE_L}{2}(e^{-i\omega_L t} - e^{i\omega_L t}) [-e\hat{r}] \cdot \hat{e}_z\]

**Expanding the dipole operator**

\[ [e\hat{r}] \cdot \hat{e}_z = I_\infty [e\hat{z}] I_\infty = \sum |nlm\rangle \langle nlm| [e\hat{z}] |n'l'm'\rangle \langle n'l'm'| \]

**Dipole matrix elements**
Interaction with classical light

Computing dipole matrix elements

\[
[e \hat{r}] \cdot \hat{e}_z = I_\infty \quad [e \hat{z}] \quad I_\infty = \sum \quad |nlm\rangle \langle nlm| \quad [e \hat{z}] \quad |n'l' m'\rangle \langle n'l' m'|
\]

\[
\langle nlm| \hat{z}|n'l' m'\rangle = \langle nlm| \left[ \int dr |r\rangle \langle r| \right] \hat{z} \left[ \int dr'| r'\rangle \langle r'| \right] |n'l' m'\rangle = \int dr \psi^*_n(lm)(r) z \psi_n(l'm')(r)
\]
Interaction with classical light

Computing dipole matrix elements

\[ [e \hat{r}] \cdot \hat{e}_z = I_\infty \langle e \hat{z} \rangle I_\infty = \sum |nlm\rangle \langle nlm| [e \hat{z}] |n' l' m'\rangle \langle n' l' m'| \]

\[ \langle nlm| \hat{z}|n' l' m'\rangle = \langle nlm| \left[ \int dr |r\rangle \langle r| \right] \hat{z} \left[ \int dr' |r'\rangle \langle r'| \right] |n' l' m'\rangle = \int dr \psi^*_{nlm}(r) z \psi_{n' l' m'}(r) \]

Example: 1s to 2p dipole transitions

\[ \langle 100| \hat{z}|21m\rangle = \int \int \int dr \, d\theta \, d\phi (r^2 \sin \theta) \left[ \frac{2}{a^{3/2}} \frac{1}{\sqrt{4\pi}} \right] z \left[ \frac{1}{8\sqrt{3}} \frac{1}{a^{3/2}} \frac{2r}{2a} e^{-r/2a} Y_{1,m}(\theta, \phi) \right] = 2a \delta_{m0} \]

Only 1s to 2p\textsubscript{z} transitions allowed
Fulfilling the resonance condition

\[ H_{\text{int}}(t) = -\hat{d}(t) \cdot \mathbf{E}(t) \propto \langle nlm | \hat{\mathcal{Z}} | n'l'm' \rangle e^{-i(E_{nlm} - E_{n'l'm'})t} e^{-i\omega_L t} + \ldots \]

Field frequency should fit energetically the energy level difference – resonance condition

The hydrogen atom

Energy levels of the hydrogen atom with some of the transitions between them that give rise to the spectral lines indicated.
Putting it all together - Roadmap

1. Shine a laser field of a given frequency (resonance condition)
Putting it all together - Roadmap

1. Shine a laser field of a given frequency (resonance condition)

2. Choose a given laser polarization (for example linearly polarized in the z direction) (selection rules for dipole-allowed transitions)
Putting it all together - Roadmap

1. Shine a laser field of a given frequency (resonance condition)

2. Choose a given laser polarization (for example linearly polarized in the z direction) (selection rules for dipole-allowed transitions)

Conclusion

\[ |100\rangle \langle 100| + |200\rangle \langle 200| + |21 - 1\rangle \langle 21 - 1| + |210\rangle \langle 210| + |211\rangle \langle 211| + |300\rangle \langle 300| + ... = \text{Unity} \]

Reduced dynamics in a two-state Hilbert space suffices!

\[ \{|100\rangle, |210\rangle\} \rightarrow \{|g\rangle, |e\rangle\} \]
The two level system

\[ |e\rangle -\]

\[ |g\rangle \]

\[ a_0 \]
The two level system

Ladder operators

\[ \sigma = |g\rangle \langle e| \quad \sigma^\dagger = |e\rangle \langle g| \]

Projectors

\[ \sigma^\dagger \sigma = |e\rangle \langle e| \quad \sigma \sigma^\dagger = |g\rangle \langle g| \]

Complete system – projectors add to unity

\[ \sigma^\dagger \sigma + \sigma \sigma^\dagger = 1 \]
Ladder operators

\[ \sigma = |g\rangle \langle e| \quad \sigma^\dagger = |e\rangle \langle g| \]

Projectors

\[ \sigma^\dagger \sigma = |e\rangle \langle e| \quad \sigma \sigma^\dagger = |g\rangle \langle g| \]

Complete system – projectors add to unity

\[ \sigma^\dagger \sigma + \sigma \sigma^\dagger = 1 \]

Free Hamiltonian

\[ H = \omega_e |e\rangle \langle e| + \omega_g |g\rangle \langle g| = \omega_e \sigma^\dagger \sigma + \omega_g (1 - \sigma^\dagger \sigma) = \omega_{eg} \sigma^\dagger \sigma + \omega_g 1 \]

Constant energy shift - ignore
The two level system

Dipole operator

\[ \hat{d} = [\langle e| + \langle g|] \hat{d} [\langle e| + \langle g|] = d_{eg} \left[ \sigma + \sigma^\dagger \right] \]
The two level system

Dipole operator

$$\hat{d} = \left| e \right\rangle \left\langle e \right| + \left| g \right\rangle \left\langle g \right| \hat{d} \left[ \left| e \right\rangle \left\langle e \right| + \left| g \right\rangle \left\langle g \right| \right] = d_{eg} \left[ \sigma + \sigma^\dagger \right]$$

Interaction in the dipole approximation

$$H_{int} = -d_{eg} \cdot \mathbf{E}(t) \left[ \sigma + \sigma^\dagger \right]$$

$$E(t) = E_0 \sin(\omega_L t) \hat{e} = \frac{iE_0}{2} (e^{-i\omega_L t} - e^{i\omega_L t}) \hat{e}.$$
The two level system

Dipole operator

\[ \hat{d} = \ket{e}\bra{e} + \ket{g}\bra{g} \hat{d} [\ket{e}\bra{e} + \ket{g}\bra{g}] = d_{eg} \left[ \sigma + \sigma^\dagger \right] \]

Interaction in the dipole approximation

\[ H_{int} = -d_{eg} \cdot E(t) \left[ \sigma + \sigma^\dagger \right] \]

\[ E(t) = E_0 \sin(\omega_L t) \hat{\epsilon} = \frac{iE_0}{2} (e^{-i\omega_L t} - e^{i\omega_L t}) \hat{\epsilon}. \]

\[ H_{int} = -\sigma \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{+i\omega_L t}) - \sigma^\dagger \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{+i\omega_L t}) \]
The two level system

**Dipole operator**

\[ \hat{d} = |e\rangle\langle e| + |g\rangle\langle g| \quad \hat{d} [|e\rangle\langle e| + |g\rangle\langle g|] = d_{eg} \left[ \sigma + \sigma^\dagger \right] \]

**Interaction in the dipole approximation**

\[ H_{\text{int}} = -d_{eg} \cdot \mathbf{E}(t) \left[ \sigma + \sigma^\dagger \right] \]

\[ E(t) = E_0 \sin(\omega_L t) \hat{\epsilon} = \frac{iE_0}{2} (e^{-i\omega_L t} - e^{i\omega_L t}) \hat{\epsilon} \]

\[ H_{\text{int}} = -\sigma \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{i\omega_L t}) - \sigma^\dagger \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{i\omega_L t}) \]

**Free operator evolution**

\[ \sigma(t) = e^{iH_0 t} \sigma e^{-iH_0 t} = \sigma e^{-i\omega_{eg} t} \]
Dipole operator

\[ \hat{d} = |e\rangle \langle e| + |g\rangle \langle g| \hat{d} [|e\rangle \langle e| + |g\rangle \langle g|] = d_{eg} \left[ \sigma + \sigma^\dagger \right] \]

Interaction in the dipole approximation

\[ H_{int} = -d_{eg} \cdot E(t) \left[ \sigma + \sigma^\dagger \right] \]

\[ E(t) = E_0 \sin(\omega_L t) \hat{\epsilon} = \frac{iE_0}{2} (e^{-i\omega_L t} - e^{i\omega_L t}) \hat{\epsilon}. \]

\[ H_{int} = -\sigma \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{i\omega_L t}) - \sigma^\dagger \frac{iE_0}{2} d_{eg} \cdot \hat{\epsilon} (e^{-i\omega_L t} - e^{i\omega_L t}) \]

Free operator evolution

\[ \sigma(t) = e^{iH_0 t} \sigma e^{-iH_0 t} = \sigma e^{-i\omega_{eg} t} \]

Fast rotating terms. Discard within the Rotating wave approximation (RWA)
Result

\[ H = \hbar \omega_{eg} \sigma^\dagger \sigma + i \Omega \left[ \sigma e^{-i \omega_L t} - \sigma^\dagger e^{i \omega_L t} \right] \]

Rabi frequency

\[ \Omega = \frac{i E_0 d_{eg} \cdot \hat{e}}{2 \hbar} \]

Approximations made on the way

- Imposing resonance and selection rules – reduction to a **two-level system**
- The **dipole approximation** (long wavelength approximation)
- Rotating wave approximation (discarding energy non-conserving terms)
HAVE A NICE WEEKEND!

(…after exercise class of course)
Maxell’s equations (no sources)
\[ \nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \]
\[ \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E}, \]
Vector potential
\[ \mathbf{E} = -\partial_t \mathbf{A} \quad \mathbf{B} = \nabla \times \mathbf{A}, \]
Coulomb gauge
\[ \nabla \cdot \mathbf{A} = 0 \]
Wave equation
\[ \nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} \quad \text{with} \quad \nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \]
Light in a box – the dipolar coupling

Wave equation

$$\nabla^2 A(r, t) = \frac{1}{c^2} \frac{\partial^2 A(r, t)}{\partial t^2}$$

Solutions

$$A^{(+)}(r, t) = i \sum_{k} c_k u_k(r) e^{-i\omega_k t}$$

$$\left[ \nabla^2 + \frac{\omega_k^2}{c^2} \right] u_k(r) = 0$$