Exercise sheet 9

for the class on 25/06/20

Excercise 1: Translation in momentum space

From the lecture notes, prove the following relation:

$$e^{ik\hat{y}}\,|p\rangle = |p+\hbar k\rangle \tag{1}$$

Excercise 2: Limit temperature in Doppler cooling

In the lecture we derived a classical equation for the damping of the atomic motion in a 1D standing wave configuration

$$\dot{p} = -\frac{2\beta}{m}p \quad \Rightarrow \quad p(t) = p(0)e^{-\frac{2\beta}{m}t}. \tag{2}$$

However this simple model neglects fluctuations of the force which give rise to heating and thus to a non-zero final temperature. To account for this, the equation of motion for the momentum operator can be supplemented by

$$\dot{p} = -\frac{2\beta}{m}p + \zeta(t), \tag{3}$$

with $\langle \zeta(t)\zeta(t')\rangle = D\delta(t-t')$, where $D = \hbar\omega_{\mathrm{rec}}\gamma$ is the diffusion coefficient. By formally integrating Eq. (3), calculate the width of the momentum distribution Δp to infer the final temperature defined by $\frac{1}{2}k_B T = \frac{\Delta p^2}{2m}$. How does it change in three dimensions?

Excercise 3: Quantum Brownian motion

The dynamics of a macroscopic oscillator (e.g. a vibrating membrane) of frequency ω_m in thermal equilibrum with the environment (temperature T) can be described by a set of Langevin equations (in terms of dimensionless quadratures):

$$\dot{q} = \omega_m p, \tag{4a}$$

$$\dot{p} = -\gamma_m p - \omega_m q + \zeta(t). \tag{4b}$$

The model includes a damping term γ_m as well as a noise term responsible for thermalization. The correlations of the noise term in the time domain are

$$\langle \zeta(t)\zeta(t')\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}\frac{\gamma_m \omega}{\omega_m}\left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)+1\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} S_{\text{th}}(\omega). \tag{5}$$

In the following we will assume that the oscillator is very good such that $\gamma_m \ll \omega_m$ and that the temperature is high compared to the mechanical frequency: $\hbar\omega_m \ll k_B T$.

a) Show that under these approximations the decay rate and thermal occupancy follow from $\gamma_m = \frac{S_{\text{th}}(\omega_m) - S_{\text{th}}(-\omega_m)}{2}$ and $\bar{n} = \frac{S_{\text{th}}(-\omega_m)}{2\gamma_m}$ (Hint: Expand $\coth(x)$ with $x = \frac{\hbar\omega}{2k_B T} \ll 1$).

b) By going to Fourier space (Definition: $A(\omega) = \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}A(t)$), show that the position quadrature is related to the mechanical susceptibility $\epsilon(\omega)$ according to $q(\omega) = \epsilon(\omega)\zeta(\omega)$ with $\epsilon(\omega) = \frac{\omega_m}{\omega_m^2 - \omega^2 - i\gamma_m\omega}$. Using this result show that the variance of position (and momentum) quadrature is given by $\langle q(t)^2\rangle = \langle p(t)^2\rangle = \bar{n} + \frac{1}{2}$.