Modern Optics
Waveguide optics

0.1 Introduction

The goal of this chapter is to give you a few of the tools necessary to understand the properties of waveguides. We will first discuss the planar waveguides for which the analytic formulation of the modes can be expressed. We will also use both a geometrical approach and one based on electromagnetism. We will then generalize the approach to cylindrical waveguide (optical fibre). At the end of this lecture we will introduce the concept of photonic crystal since it is the cornerstone for one type guidance microstructured fibre, which were introduced about 25 years ago by Prof. Philip Russell.

0.2 Planar waveguide

0.2.1 Geometrical approach

In this approach we define the waveguide as shown on Fig. 1. It typically consists of one material with a refractive index n_1 embedded in another material with a lower refractive index n_2. A for its dimension, the waveguide is infinite in $y-$direction and has a thickness $2a$. The light propagates along the $z-$direction. Note that geometrical approach is valid as long as $2a \gg \lambda$ where λ is the wavelength of the propagating light.

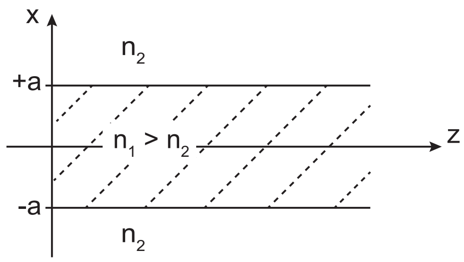

Figure 1: Schematics of a planar waveguide

As we know from Snell-Descartes' law, when the light arrives at the surface between n_1 and n_2 it will experience total internal reflection if the incident angle is larger than the critical angle $i_c = \text{asin}\left({}^{n_2}/_{n_1}\right)$. Note that rather than measuring the incident angle from the normal to the surface, we use the angle θ as defined on Fig. 2. With such definition the critical angle is now defined as

$$\cos\theta_c = \frac{n_2}{n_1} \tag{1}$$

We can then define the propagation constant

$$\beta = k_0 n_1 \cos\theta \tag{2}$$

where $k_0 = {}^{\omega}/_{c}$ is the wavenumber in vacuum. As we can see this is simply the projection of the wavevector $\mathbf{k}$ along the direction of propagation (Oz).

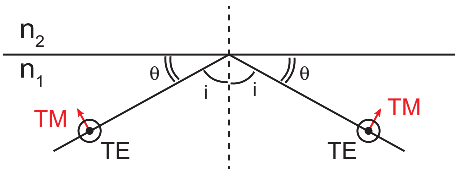

Figure 2: Definitions of angles above the critical angle of incidence i_c.

Guidance conditions

To get guidance in the waveguide two conditions must be fulfilled. First of all the angle of incidence has to be $\theta < \theta_c$ to insure total internal reflection at the boundary. Note that because of eq. (1) we have

$$0 < \theta < \theta_c \Rightarrow 1 > \cos\theta > \cos\theta_c = \frac{n_2}{n_1}$$
$$\Rightarrow \; k_0 n_1 > \beta > k_0 n_2 \tag{3}$$

Resonance conditions

Let assume that a wave is propagating inside the waveguide (Fig. 3). The resonance conditions applies between to planes Σ_1 and Σ_2. In order to associate a plane wave to the cay CD, we must have certain conditions on θ_i since both front Σ_1 and Σ_2 must be in phase. The optical path between C and D must lead to the same oscillating state. The waveguide can them be seen as a **resonator**. The total dephasing is

$$\Delta\phi = 2\phi_{\text{refl.}} + k_0 n_1 (CD - AB) = 2p\pi \quad \text{with } p \in \mathbb{Z}^* \tag{4}$$

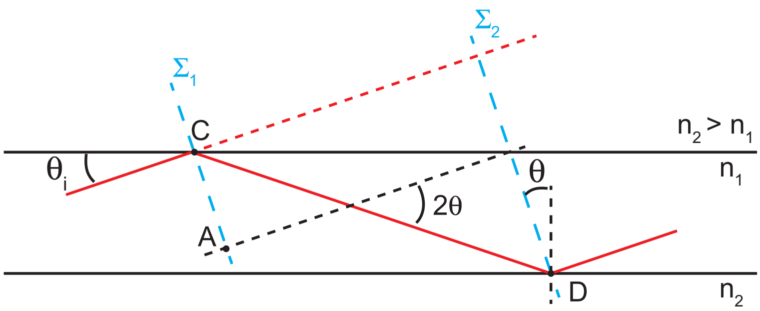

Figure 3: Schematics of a beam propagating inside the planar waveguide.

A mode of the waveguide must fulfill <u>both</u> conditions and due to the resonance conditions, not every value of $\theta \in [0, \theta_c[$ will correspond to a mode. There are actually discrete values for θ.

$$\boxed{\text{Guided mode :} \quad k_0 n_1 > \beta > k_0 n_2 \;\; \underline{\text{and}} \; \Delta\phi = 2p\pi \text{ with } p \in \mathbb{Z}^*} \tag{5}$$

Note that we know the dephasing introduced at the reflection (see chapter 1). This depends on the polarisation of the incident beam:

$$\phi_{\text{TE}} = -2 \text{ atan}\left(\sqrt{\frac{\beta^2 - \mathrm{k}_0^2 \mathrm{n}_2^2}{\mathrm{n}_1^2 \mathrm{k}_0^2 - \beta^2}}\right) \tag{6a}$$

$$\phi_{\text{TM}} = -2 \text{ atan}\left(\frac{\mathrm{n}_1^2}{\mathrm{n}_2^2}\sqrt{\frac{\beta^2 - \mathrm{k}_0^2 \mathrm{n}_2^2}{\mathrm{n}_1^2 \mathrm{k}_0^2 - \beta^2}}\right) \tag{6b}$$

Guided solutions

Since we have introduced the longitudinal wavevector β we can also introduce the transverse propagation constant α

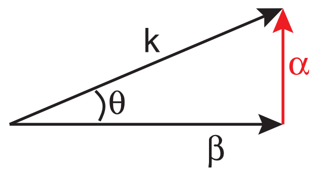

Figure 4: Definition of propagation constants

The transverse propagation constant is simply

$$\alpha = k_0 n_1 \sin\theta = \sqrt{k_0^2 n_1^2 - \beta^2} \tag{7}$$

Note that since we defined α and *beta* for the core region (n_1) we can also defined the longitudinal propagation constant in the cladding region (n_2) by

$$\gamma = \sqrt{\beta^2 - k_0^2 n_2^2} \tag{8}$$

It is important to notice that α, *beta* and γ are not independent to each other:

$$\alpha^2 + \beta^2 = k_0^2 n_1^2 \tag{9}$$
$$\alpha^2 + \gamma^2 = k_0^2 (n_1^2 - n_2^2) \tag{10}$$

It is a common practise to use the dimension of the waveguide in order to normalize these parameters. We then introduce U, V and W such that

$$\left.\begin{array}{rcl} U & = & a \times \alpha \\ \\ W & = & a \times \gamma \end{array}\right\} U^2 + W^2 = V^2 = k_0^2 a^2 (n_1^2 - n_2^2) \tag{11}$$

As we see the parameter V, which is called the *waveguide parameter* only depends on the opto-geometrical properties of the waveguide (a the size of the waveguide, and (n_1, n_2) the refractive indices of the materials, which compose the waveguide).

Solving the problem

Using Fig. 3 we can write

$$\left\{\begin{array}{rcl} CD & = & \dfrac{2a}{\sin\theta} \\ \\ AB & = & CD\cos 2\theta = \dfrac{2a}{\sin\theta}(\cos^2\theta - \sin^2\theta) \end{array}\right. \tag{12}$$

We can then calculate the optical path difference

$$\begin{aligned} \delta = n_1(CD - AB) &= n_1\left[\frac{2a}{\sin\theta} - \frac{2a}{\sin\theta}(\cos^2\theta - \sin^2\theta)\right] \\ &= \frac{2an_1}{\sin\theta}(2\sin^2\theta) = 4an_1\sin\theta \end{aligned} \tag{13}$$

It is now possible to evaluate the total phase shift, depending of the considered polarization.

TE mode: In that case, we already gave the phase shift introduced at the reflection of the beam at the interface between n_1 and n_2. Considering the definition of α, β and γ, this phase shift is simply given by

$$\phi_{\mathrm{TE}} = -2\mathrm{atan}\left(\frac{\gamma}{\alpha}\right) \tag{14}$$

yielding the total phase-shift

$$\Delta\phi = -4\mathrm{atan}\left(\frac{\gamma}{\alpha}\right) + 4\mathrm{ak_0n_1}\sin\theta = -4\mathrm{atan}\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) + 4\mathrm{a}\alpha = 2\mathrm{p}\pi \text{ with } \mathrm{p} \in \mathbb{Z} \tag{15}$$

At this stage it is important to distinguish when the integer number p is even or odd.

Even modes: In that case we can write $p = 2n$ where n is an integer number. In this case the eq. (15) simply becomes

$$\begin{aligned} \Delta\phi &= -4\mathrm{atan}\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) + 4\alpha\mathrm{a} = 4\mathrm{n}\pi \\ \Leftrightarrow \ \alpha a - n\pi &= \mathrm{atan}\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) \quad \tan(\alpha\mathrm{a} - \mathrm{n}\pi) = \tan(\alpha\mathrm{a}) = \tan\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) \end{aligned} \tag{16}$$

Finally, we obtain an eigenvalue equation

$$W = U\tan U \tag{17}$$

Of course, since tan is a periodic function there should be more than one solution to this equation. Each of these solutions corresponds to one value of p.

Odd mode of course we need to look at the other set of modes, corresponding to $p = 2q+1$ where q is an integer number. Once again we can evaluate the total phase shift

$$\begin{aligned} \Delta\phi &= -4\mathrm{atan}\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) + 4\alpha\mathrm{a} = 2(2\mathrm{q}+1)\pi = 4\mathrm{q}\pi + 2\pi \\ \Leftrightarrow \ \mathrm{atan}\left(\frac{\gamma\mathrm{a}}{\alpha\mathrm{a}}\right) &+ \alpha\mathrm{a} = \mathrm{q}\pi + \frac{\pi}{2} \end{aligned} \tag{18}$$

and therefore we obtain the eigenvalue equation

$$W = -U\mathrm{cotan}\ U \tag{19}$$

Summary for the TE mode:

$$\boxed{\begin{array}{lll} \text{TE mode} & \text{even mode} & W = U\tan U \\ & \text{odd mode} & W = -U\mathrm{cotan}\ U \\ & & \\ \multicolumn{3}{l}{\text{and } U^2 + W^2 = V^2 = k_0^2 a^2 (n_1^2 - n_2^2)} \end{array}} \tag{20}$$

Summary for TM mode: Although we have not really discuss the TM mode, the calculation can be done by following the same steps. In that case the eigenvalue equations for respectively the even and odd mode are

$$\boxed{\begin{array}{lll} \text{TM mode} & \text{even mode} & n_1^2\ W = n_2^2\ U\tan U \\ & \text{odd mode} & n_1^2\ W = -n_2^2\ U\mathrm{cotan}\ U \\ & & \\ \multicolumn{3}{l}{\text{and } U^2 + W^2 = V^2 = k_0^2 a^2 (n_1^2 - n_2^2)} \end{array}} \tag{21}$$

How to solve the problem? There is unfortunately no trivial solution for the equation (20) and a numerical approach is necessary. Since the waveguide parameter only depends on the opto-geometrical properties of the waveguide and is linked to α and γ we can graphically solve the problem by plotting W as a function of U. In such a plot the waveguide parameter is a simple circle with a radius V.

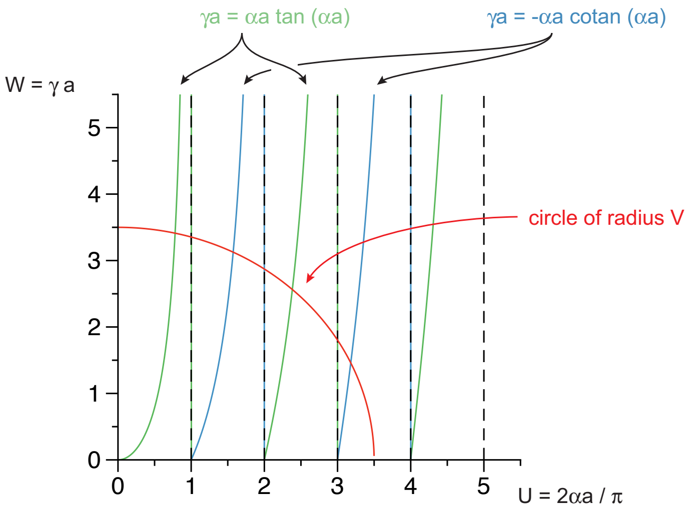

Figure 5: Solving graphically the eigenvalue problem for a planar waveguide.

We can also notice that

$$\left.\begin{array}{rcl}\alpha & = & k_0 n_1 \sin\theta > 0 \\ \gamma & = & \sqrt{\beta^2 - k_0^2 n_2^2} > 0\end{array}\right\} \Rightarrow \operatorname{atan}\left(\frac{\gamma a}{\alpha a}\right) \in \left[0; \frac{\pi}{2}\right[\tag{22}$$

Therefore

$$\left\{\begin{array}{llll}\text{for even modes} & -\operatorname{atan}\left(\frac{\gamma a}{\alpha a}\right) + \alpha a = p\pi & \rightarrow & p\pi < \alpha_p a < \left(p + \frac{1}{2}\right)\pi \\ \text{for odd modes} & p = (2q+1) & \rightarrow & (2q+1)\pi < \alpha_{2q+1} a < \left(2q + \frac{3}{2}\right)\pi\end{array}\right. \tag{23}$$

As we see from the plot on Fig. 5, we have only a discrete number of solution for the problem. In fact every $\pi/2$ there is one unique solution. Since the extreme value is set by V the problem has

$$\frac{2V}{\pi} \text{ solutions} \tag{24}$$

Consequences of the multimodeness

For a given mode the only possibility to change the number of excitable modes is by changing the wavelength. This then modifies the factor k_0 in eq. (20), (21). This of course can lead to several consequences depending on how flexible is the used wavelength:

1. Since the phase velocity is

$$v_\varphi = \frac{\omega}{\beta_p} \tag{25}$$

and β_p obviously depends on the index p of the chosen solution, each mode p will experience a different phase velocity. This results into *inter-modal dispersion*. Note that as in a material the phase velocity depends on the refractive index ($v_\varphi = {}^c/_n$) here we define an *effective* refractive index $n_{\text{eff,p}}$ by analogy. The phase velocity can then be written as

$$v_\varphi = \frac{c}{\beta_p} = \frac{c}{n_1 \cos\theta_p} = \frac{c}{n_{\text{eff,p}}} \tag{26}$$

2. As we can see from Fig. 5 there is always one solution: at least one mode can always be excited. Note that this is only true because we choose a symmetric waveguide. In the case of a non-symmetric waveguide ($n_1/n_2/n_3$) there could domain of parameter when no mode can be excited.

3. The large γ (or W) the smaller the index p of the mode. Moreover the larger γ the smaller α. And since $\alpha = k_0 n_1 \sin\theta$ this corresponds to a very small angle θ. From eq. (26) we see that in this case the effective index gets closer to the one of the core region n_1. The fundamental mode is the mode with the largest possible β.

4. By contrast if θ gets larger, and closer to the critical angle θ_c the effective index $n_{\text{eff,p}} \to n_2$.

5. **Cut-off frequency** Let assume that we want to guide the light from a laser (its frequency is not determined yet – $\omega \in [0, +\infty[$) and in a waveguide. The angle is $\theta \in [0, \theta_c[$. How to determine the limit for the frequency if the mode n is desired? Actually, this simply means that the radius of the circle described by V (Fig. 5 is large enough to include the frequency $n \times ({}^\pi/_2)$. From the definition of the waveguide parameter $V = ({}^\omega/_c)\, a\sqrt{n_1^2 - n_2^2}$ we deduce

$$\omega \geq \frac{n\pi c}{2a\sqrt{n_1^2 - n_2^2}} \tag{27}$$

Of course we can use the same argument to estimate the largest possible frequency for which the waveguide is single-mode:

$$\omega \leq \frac{\pi c}{2a\sqrt{n_1^2 - n_2^2}} \tag{28}$$

This special frequency is called the **cut-off frequency**.

0.2.2 Electromagnetic approach

The idea here is now to use Maxwell's equation in order to derive the eigenvalue of the waveguide. Not only we should be able to retrieve the same result as eq. (20), (21) but this should allow us to express the electric and the magnetic field of the modes.

Equations of propagation

The materials that are used for waveguide are dielectric and do not have any charge nor magnetization ($\rho = 0, \mathbf{M} = \mathbf{0}$). We will also assume that the material is lossless ($\mathbf{J} = 0$). Within these approximations the Maxwell's equations write

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{29a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{29b}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{29c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{29d}$$

and the constitutive equations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon_0 n^2 \mathbf{E} \tag{30a}$$
$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu_0 \mathbf{H} \tag{30b}$$

Using the algebraic equations

$$\nabla \cdot (\alpha \mathbf{A}) = \alpha \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \alpha \tag{31a}$$
$$\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{31b}$$

and the constitutive equations we have

$$\begin{aligned}\nabla \cdot \mathbf{D} = \nabla \cdot \left(\epsilon_0 n^2 \mathbf{E}\right) &= \epsilon_0 n^2 \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \left(\epsilon_0 n^2\right) = 0 \\ \Rightarrow \quad \nabla \cdot \mathbf{E} &= \frac{-1}{\cancel{\epsilon_0}\, n^2} \mathbf{E} \cdot \nabla \left(\cancel{\epsilon_0}\, n^2\right)\end{aligned} \tag{32}$$

Of course in this equation we can cancel the term ϵ_0, which is a constant, but not n^2 since it can be anything and varies... Using eq. (**??**) yields

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ &= -\mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\mu_0 \epsilon_0 n^2 \frac{\partial^2 \mathbf{E}}{\partial t} \quad \text{since } n \neq n(t)\end{aligned} \tag{33}$$

And finally we have the propagation equation for the electric field in the dielectric

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 n^2 \frac{\partial^2 \mathbf{E}}{\partial t} = \frac{1}{\epsilon} \nabla \left[\mathbf{E} \cdot \nabla \epsilon\right] \tag{34}$$

Here we used $\epsilon = \epsilon_0 n^2$. Note that we usually assume that the refractive is constant and therefore that $\nabla n^2 = 0$. In a waveguide we need at least two different refractive indices. In the core we can also have a gradual change of the refractive index (gradient-index optical fiber). This leads to the extra term on the right-hand side the equation (34). We can also derive the equation for the propagation of the magnetic field[1] $\mathbf{H}$:

$$\nabla^2 \mathbf{H} - \epsilon \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t} = 0 \tag{35}$$

planar-waveguide

The aim here is to solve the eq. (34) and (35) for the planar waveguide. As previously we define the planar waveguide as shown on Fig. 6. Of course in the direction (Ox) the term $\nabla \epsilon$ is not null. However in the case of TE mode since the electric field only has a component along (Oy) the scalar product $\mathbf{E} \cdot \nabla \epsilon = 0$. For TM modes the situation is more complex.

Let consider a TE mode

$$\mathbf{E} = \begin{pmatrix} 0 \\ E_0(x) \exp i (\omega t - \beta z) \\ 0 \end{pmatrix} \tag{36}$$

[1] Of course this time the right-hand side of the equation is null since there is no magnetisation.

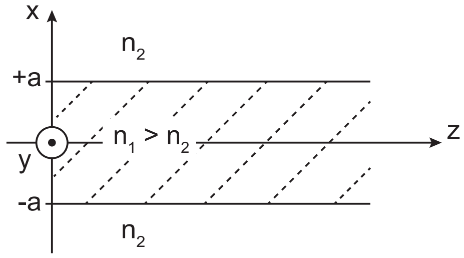

Figure 6: Schematics of the planar waveguide

where β is the constant of propagation of the mode and is independent of the coordinated (x, y). From Faraday equation expressed using the complex form we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -i\omega\mu_0 \mathbf{H} \;\Rightarrow \mathbf{H} = \frac{i}{\mu_0 \omega} \nabla \times \mathbf{E} \tag{37}$$

from which we can extract the magnetic field

$$\mathbf{H} = \begin{pmatrix} \left(-\beta/\mu_0\omega\right) E_0(x) \exp i(\omega t - \beta z) \\ 0 \\ \left(i/\mu_0\omega\right) \dfrac{\partial E_0(x)}{\partial x} \exp i(\omega t - \beta z) \end{pmatrix} \propto \begin{pmatrix} -\partial_z E_0(x) \\ 0 \\ \partial_x E_0(x) \end{pmatrix} \tag{38}$$

Note that the magnetic field is not transverse!

Let's assume that the waveguide is symmetric. We can define it by

$$\begin{cases} \text{core region} & \rightarrow \quad x \in [-a, +a] \\ \text{cladding region} & \rightarrow \quad |a| > a \end{cases} \tag{39}$$

For simplicity we consider the TE mode for which $\epsilon \cdot \nabla \epsilon = 0$ in the propagation equation (eq. (34)) and the field is defined by eq. (36). Inserting the electric field in the propagation equation we have

$$\nabla^2 \mathbf{E} \rightarrow \partial_{xx} E_0(x) \exp i(\omega t - \beta z) - \beta^2 E_0(x) \exp i(\omega t - \beta z) \tag{40}$$

$$\partial_{tt} \mathbf{E} \rightarrow -\omega^2 E_0(x) \exp i(\omega t - \beta z) \tag{41}$$

and inserting it into the propagation equation (eq. (34)) yields

$$\partial_{xx} E_0(x) \exp i(\omega t - \beta z) - \beta^2 E_0(x) \exp i(\omega t - \beta z) + \mu_0 \epsilon \omega^2 E_0(x) \exp i(\omega t - \beta z) \tag{42}$$

This equation is very general, and we should distinguish whether we are considering the core ($\mu_0 \epsilon \omega^2 = k_0^2 n_1^2$) or the cladding region ($\mu_0 \epsilon \omega^2 = k_0^2 n_2^2$):

$$\text{in the core:} \quad \partial_{xx} E_0(x) + \left(k_0^2 n_1^2 - \beta^2\right) E_0(x) = 0 \tag{43}$$

$$\text{in the cladding:} \quad \partial_{xx} E_0(x) + \left(k_0^2 n_2^2 - \beta^2\right) E_0(x) = 0 \tag{44}$$

Looking at guided solution imposes that the solution are exponentially decaying in the cladding but not in the core. To insure this we need $k_0^2 n_1^2 - \beta^2 > 0$ and $k_0^2 n_2^2 - \beta^2 < 0$. Combined together this yields the condition on β for guided solutions[2]:

$$\boxed{k_0 n_2 \leq \beta \leq k_0 n_1} \tag{45}$$

which the same condition as the one we found geometrically (eq. (5)).

[2] if $0 \leq \beta \leq k_0 n_2$ we have modes in the cladding.

Guided solutions Because of the guidance conditions (eq. (45), we can rewrite the equations of propagation by introducing the parameters $\alpha^2 = k_0^2 n_1^2 - \beta^2$ and $\gamma^2 = \beta^2 - k_0^2 n_1^2$ so that the equations of propagation are

$$\text{in the core: } \partial_{xx} E_0(x) + \alpha^2 E_0(x) = 0 \tag{46}$$
$$\text{in the cladding: } \partial_{xx} E_0(x) - \gamma^2 E_0(x) = 0 \tag{47}$$

The solution for eq. (46) is oscillating : $E_0(x) = C\cos\alpha x$ or $E_0(x) = C\sin\alpha x$ depending whether we consider an even or an odd mode. On the other hand in the cladding the solution has the form $E_0(x) = C\exp -\gamma|x|$, where C is a constant value to be determined. For the different region of the waveguide we then have

$$\text{even mode: } \left\{ \begin{array}{lcll} E_0(x) & = & Ce^{-\gamma x} & \text{for } x > a \\ E_0(x) & = & C'\cos\alpha x & \text{for } -a < x < +a \\ E_0(x) & = & C\text{"}e^{\gamma x} & \text{for } x < a \end{array} \right. \tag{48}$$

$$\text{odd mode: } \left\{ \begin{array}{lcll} E_0(x) & = & Ce^{-\gamma x} & \text{for } x > a \\ E_0(x) & = & C'\sin\alpha x & \text{for } -a < x < +a \\ E_0(x) & = & C\text{"}e^{\gamma x} & \text{for } x < a \end{array} \right. \tag{49}$$

Case of the even mode: Since we are looking at TE mode $\mathbf{E} = E_0(x)\widehat{\mathbf{e_y}}$. Moreover because of the definition of the field and the waveguide (Fig. 6, the boundary conditions only affect E_y and H_z. From eq. (38) we see that the z-component of the magnetic field is simply proportional to $\partial_x E_0(x)$.

Therefore we have the electric and the magnetic fields in the different regions of the waveguide:

$$\begin{array}{rlcl|lcl} \text{for } x > a & E_0(x) & = & Ce^{-\gamma x} & \partial_x E_0(x) & = & -\gamma C e^{-\gamma x} \\ \text{for } -a < x < a & E_0(x) & = & C'\cos\alpha x & \partial_x E_0(x) & = & -\gamma C'\sin\alpha x \\ \text{for } x < -a & E_0(x) & = & C''e^{\gamma x} & \partial_x E_0(x) & = & \gamma C'' e^{\gamma x} \end{array} \tag{50}$$

Obviously we need to apply the boundary conditions at the interface between the core and the cladding at $r = a$:

$$\begin{array}{lrcl|rcl} \text{at } x = +a & Ce^{-\gamma a} & = & C'\cos\alpha a & -\gamma Ce^{-\gamma a} & = & -\alpha C'\sin\alpha a \\ \text{at } x = -a & C''e^{-\gamma a} & = & C'\cos\alpha a & \gamma C''e^{-\gamma a} & = & +\alpha C'\sin\alpha a \end{array} \tag{51}$$

Moreover for a symmetric waveguide we can simplify these equations by setting that $C \equiv C''$. We then have a set of three equations with three unknown:

$$\left\{ \begin{array}{ccccc} Ce^{-\gamma a} & - & C'\cos\alpha a & & = 0 \\ & - & C'\cos\alpha a & + \; C''e^{-\gamma a} & = 0 \\ -C\gamma e^{-\gamma a} & + & C'\alpha\sin\alpha a & & = 0 \end{array} \right. \tag{52}$$

which has a solution only if the determinant is null:

$$\begin{aligned} \det(\dots) &= e^{-\gamma a}\left(-\alpha\sin\alpha a e^{-\gamma a}\right) - \gamma e^{-\gamma a}\left(-e^{-\gamma a}\cos\alpha a\right) = 0 \\ &\Longleftrightarrow -\alpha\sin\alpha a + \gamma\cos\alpha a = 0 \\ &\Longrightarrow \frac{\gamma}{\alpha} = \tan\alpha a \end{aligned} \tag{53}$$

This yields the eigenvalue equation:

$$\boxed{\gamma a = \alpha a \tan \alpha a} \tag{54}$$

Which is the very same that we found by the geometrical approach. We could however do a few comments:

1. I principle the boundary conditions (eq. (51) lead to four equations. Considering the eigenvalue (eq. (54)), we see that the last equation

$$\alpha\gamma C'' e^{\gamma a} = \alpha a C' \sin \alpha a$$

can simply be reduced to

$$\gamma a C'' e^{\gamma a} = \frac{\gamma a}{\tan \alpha a} C' \sin \alpha a = \frac{\gamma a}{\cancel{\sin \alpha a}} \cos \alpha a \; C' \cancel{\sin \alpha a}$$

$$\iff \quad C'' e^{\gamma a} = C' \cos \alpha a \tag{55}$$

which is redundant with the second equation.

2. If we use the definition of the electric field $E(x)$ in the propagation equation we get for the core and the cladding:

$$\forall x > a \qquad \left.\begin{array}{rcl} E(x) & = & Ae^{-\gamma x} \\ \partial_{xx} E + \left(k_0^2 n_2 - \beta^2\right) E(x) & = & 0 \end{array}\right\} \Longrightarrow \gamma^2 - \beta^2 + k_0^2 n_2^2 = 0$$

$$\forall x \in [-a, a] \qquad \left.\begin{array}{rcl} E(x) & = & B \cos \alpha a \\ \partial_{xx} E - \left(\beta^2 - k_0^2 n_2\right) E(x) & = & 0 \end{array}\right\} \Longrightarrow \alpha^2 + \beta^2 = k_0^2 n_1^2$$

yielding

$$\boxed{\alpha^2 + \gamma^2 = k_0^2 \left(n_1^2 - n_2^2\right)} \tag{56}$$

which, again, is the same equation that we established by the geometrical approach.

Modal field components for even TE mode As we see we have to solve the same eigenvalue equation that we derived by using the geometrical approach. Obviously we can use the same technique as explained before. The advantage is that now the knowledge of β and consequently α and γ can give the full electromagnetic structure of the field propagating in the waveguide by using eq. (50). This is summarized thereafter:

	e_y	h_x	h_z
core	$\dfrac{\cos(Ux)}{\cos U}$	$\dfrac{\beta}{k}\left(\dfrac{\epsilon_0}{\mu_0}\right)^2 \dfrac{\cos(Ux)}{\cos U}$	$\dfrac{iW}{ka}\left(\dfrac{\epsilon_0}{\mu_0}\right)^2 \dfrac{\sin(Ux)}{\sin U}$
cladding	$\dfrac{\exp(-W\|x\|)}{\exp(-W)}$	$\dfrac{-\beta}{k}\left(\dfrac{\epsilon_0}{\mu_0}\right)^2 \dfrac{\exp(-W\|x\|)}{\exp(-W)}$	$\dfrac{iW}{ka}\left(\dfrac{\epsilon_0}{\mu_0}\right)^2 \dfrac{x}{\|x\|} \dfrac{\exp(-W\|x\|)}{\exp(-W)}$

The other components (e_x, e_z, h_y) are null. Of course what was done here in detail for the even TE mode can be repeated for the odd TE mode or the TM modes (odd and even).

General case

As a general feature we can decompose the field of the mode j into the transverse and longitudinal components

$$\mathbf{E}_j = [\mathbf{e}_{t,j}(x,y) + \mathbf{e}_{z,j}(x,y)\hat{\mathbf{z}}] \exp(i\beta_j z) \tag{57a}$$
$$\mathbf{H}_j = [\mathbf{h}_{t,j}(x,y) + \mathbf{h}_{z,j}(x,y)\hat{\mathbf{z}}] \exp(i\beta_j z) \tag{57b}$$

In this form it can be shown that for non-absorbing waveguide, where $n(x,y)$ is real it is possible to find the field components of each bound mode such that they satisfy

$$\mathbf{e}_{t,j}, \mathbf{h}_{t,j} \text{ purely real}; \qquad \mathbf{e}_{z,j}, \mathbf{h}_{z,j} \text{ purely imaginary} \tag{58}$$

Moreover the *vector* Laplacian ∇^2 considerably complicates the problem since it couples component of the field vector. If the situation is simplified in Cartesian coordinates, where the vector Laplacian is simply a *scalar* operator, it will not be the case in other system of coordinate, such as in cylindrical coordinate which is used for optical fibre. Without entering the details we can show that the general form of the propagation for the electromagnetic field is given by

$$\left(\nabla_\perp^2 + n^2k^2 - \beta_j^2\right)\mathbf{e}_j = -\left(\nabla_\perp + i\beta_j\hat{\mathbf{z}}\right)\left(\mathbf{e}_{t,j}\cdot\nabla_\perp \ln n^2\right) \tag{59a}$$
$$\left(\nabla_\perp^2 + n^2k^2 - \beta_j^2\right)\mathbf{h}_j = -\left(\nabla_\perp \ln n^2\right) \times \{(\nabla_\perp + i\beta_j\hat{\mathbf{z}}) \times \mathbf{h}_{t,j}\} \tag{59b}$$

from which we can deduce the transverse components as long as the longitudinal components $e_{z,j}$ and $h_{z,j}$ of the j^{th} mode is known:

$$\mathbf{e}_{t,j} = \frac{i}{k^2n^2(x,y) - \beta_j^2}\left[\beta_j\nabla_\perp\mathbf{e}_{z,j} - \left(\frac{\epsilon_0}{\mu_0}\right)^2 k\ \hat{\mathbf{z}} \times \nabla_\perp\mathbf{h}_z\right] \tag{60a}$$
$$\mathbf{h}_{t,j} = \frac{i}{k^2n^2(x,y) - \beta_j^2}\left[\beta_j\nabla_\perp\mathbf{h}_{z,j} + \left(\frac{\epsilon_0}{\mu_0}\right)^2 k\ n^2(x,y)\hat{\mathbf{z}} \times \nabla_\perp\mathbf{e}_z\right] \tag{60b}$$

where $k = ({}^{2\pi}/_{\lambda})$ is the wavenumber and the refractive index is $n = n(x,y)$. And to obtain the longitudinal component we can show that these satisfy a pair of coupled equations:

$$\left(\nabla_\perp^2 + p_j\right)e_{z,j} - \frac{\beta_j^2}{p_j}\nabla_\perp e_{z,j}\cdot\nabla_\perp \ln n^2 = -\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{k\beta_j}{p_j}\hat{\mathbf{z}}\cdot\left(\nabla_\perp h_{z,j} \times \nabla_\perp \ln n^2\right) \tag{61a}$$
$$\left(\nabla_\perp^2 + p_j\right)h_{z,j} - \frac{n^2k^2}{p_j}\nabla_\perp h_{z,j}\cdot\nabla_\perp \ln n^2 = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{kn^2\beta_j}{p_j}\hat{\mathbf{z}}\cdot\left(\nabla_\perp e_{z,j} \times \nabla_\perp \ln n^2\right) \tag{61b}$$
$$\tag{61c}$$

where $p_j = n^2k^2 - \beta_j^2$ and $n = n(x,y)$. As we can see, the presence of the $\nabla_\perp \ln n^2$ complicates significantly the problem...

0.3 Optical fibre

0.3.1 step-index fibre

A step-index fiber is schematically represented on Fig. 7. For such fibres the refractive index of the core region is supposed to be constant. To remain consistent with the previous section it is called n_2. The refractive of the cladding n_1 is also constant. To allow guidance in the core the refractive index in the core has to be larger than the cladding one. For telecommunication fibres are made of silica and the refractive index is modified by doping either the core with Germanium, in order to increase its refractive index ($n_1 > n_2$) or the cladding with Fluor or potassium in order to lower its refractive index ($n_2 < n_1$). The

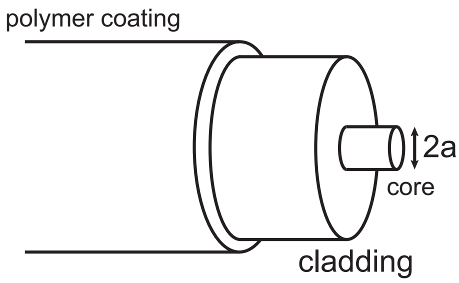

Figure 7: step-index fibre. The core has a radius a and a refractive index n_1. The cladding has a refractive index n_2. The polymer coating acts as a protection for the fibre as well as a way to eliminate the light leaking out of the core.

conventional fibre for telecommunication has a core diameter of $\sim 8.2\mu$m and an external diameter of 125μm. The *profile height parameter*

$$\Delta n = \frac{1}{2}\left(1 - \frac{n_{co}^2}{n_{cl}^2}\right) \simeq \frac{n_1 - n_2}{n_1} \simeq 5 \times 10^{-3} \tag{62}$$

. Although this is a small value it is enough to guide light very efficiently. The best transmission loss figures so far is 0.154 dB/km.
The goal here is

1. Establish the eigenvalue equation in order to evaluate the propagation constant β.
2. Distinguish the different possible modes (TE/TM/EH/HE...etc).
3. Go beyond the step-index model and discuss the possible other type of mechanisms, especially in microstructured fibres.

Finding the optical modes. The eigenvalue problem

Since we have cylindrical symmetry we need to express the electromagnetic field as

$$\mathbf{E} = \mathbf{e}(r,\theta)e^{i\omega t - \beta z} \tag{63a}$$

$$\mathbf{H} = \mathbf{h}(r,\theta)e^{i\omega t - \beta z} \tag{63b}$$

and the Laplacian in the Helmholtz equation now leads to the propagation equations

$$\partial_{rr}e_z + \frac{1}{r}\partial_r e_z + \frac{1}{r^2}\partial_{\theta\theta}e_z + \left[k^2 n(r,\theta)^2 - \beta^2\right] e_z = 0 \tag{64a}$$

$$\partial_{rr}h_z + \frac{1}{r}\partial_r h_z + \frac{1}{r^2}\partial_{\theta\theta}h_z + \left[k^2 n(r,\theta)^2 - \beta^2\right] h_z = 0 \tag{64b}$$

Note also that in that equation the refractive index can simply be written as $n(r,\theta) = n(r)$ because of the symmetry of the system.

From Maxwell equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{65a}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{65b}$$

$$\tag{65c}$$

where $\epsilon = \epsilon_0 n^2$ with n the refractive index of the material. Using the operator $\nabla = \left(\frac{\partial}{\partial r}; \frac{1}{r}\frac{\partial}{\partial \theta}; \frac{\partial}{\partial z}\right)$ we have the different components of the electric field

$$\frac{1}{r}\frac{\partial e_z}{\partial \theta} + i\beta e_\theta = -i\omega\mu_0 h_r \tag{66a}$$

$$-i\beta e_r - \frac{\partial e_z}{\partial r} = -i\omega\mu_0 h_\theta \tag{66b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(re_\theta) - \frac{1}{r}\frac{\partial e_r}{\partial \theta} = -i\omega\mu_0 h_z \tag{66c}$$

and the magnetic field

$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} + i\beta h_\theta = +i\omega\epsilon_0 n^2 e_r \tag{67a}$$

$$-i\beta h_r - \frac{\partial h_z}{\partial r} = +i\omega\epsilon_0 n^2 e_\theta \tag{67b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rh_\theta) - \frac{1}{r}\frac{\partial h_r}{\partial \theta} = +i\omega\epsilon_0 n^2 e_z \tag{67c}$$

From eq. (66b) and (67a) we can extract h_θ

$$-i\beta e_r - \frac{\partial e_z}{\partial r} = -i\omega\mu_0 h_\theta \qquad \times\, i\beta \tag{68a}$$

$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} + i\beta h_\theta = i\omega\epsilon_0 n^2 e_r \qquad \times\,(-i\omega\mu_0) \tag{68b}$$

$$\Rightarrow \quad -i\left(\beta\frac{\partial e_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial \theta}\right) = \left(\omega^2\epsilon_0 n^2\mu_0 - \beta^2\right) e_r \tag{69}$$

Similarly we can express the different transverse components of the electric and the magnetic field as a function of the longitudinal part of the field e_z and h_z:

$$e_r = \frac{-i}{k^2n^2 - \beta^2}\left(\beta\frac{\partial e_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial \theta}\right) \tag{70a}$$

$$e_\theta = \frac{-i}{k^2n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial e_z}{\partial \theta} - \omega\mu_0\frac{\partial h_z}{\partial r}\right) \tag{70b}$$

$$h_r = \frac{-i}{k^2n^2 - \beta^2}\left(\beta\frac{\partial h_z}{\partial r} - \frac{\omega\epsilon_0 n^2}{r}\frac{\partial e_z}{\partial \theta}\right) \tag{70c}$$

$$h_\theta = \frac{-i}{k^2n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial h_z}{\partial \theta} + \omega\epsilon_0 n^2\frac{\partial e_z}{\partial r}\right) \tag{70d}$$

fibre modes

There exist three types of modes depending on the value of e_z and h_z:

1. if $e_z = 0$ we talk about TE mode.
2. if $h_z = 0$ we talk about TM mode.
3. if neither e_z nor h_z is null then the mode is called an *hybrid mode*.

TE modes They correspond to the case when $e_z = 0$. This obviously strongly simplify the set of eq.(70)

$$e_r = \frac{-i\omega\mu_0}{k^2n^2-\beta^2}\frac{1}{r}\frac{\partial h_z}{\partial\theta} \tag{71a}$$

$$e_\theta = \frac{i\omega\mu_0}{k^2n^2-\beta^2}\frac{\partial h_z}{\partial r} \tag{71b}$$

$$h_r = \frac{-i\beta}{k^2n^2-\beta^2}\frac{\partial h_z}{\partial r} \tag{71c}$$

$$h_\theta = \frac{-i\beta}{k^2n^2-\beta^2}\frac{1}{r}\frac{\partial h_z}{\partial\theta} \tag{71d}$$

but also the propagation equations ((64)). Indeed since $e_z = 0$ we only need to deal with the propagation equation applied to the magnetic field **H** (eq. (64b). Moreover in order to take the cylindrical symmetry of the problem the magnetic field must be expressed as a function of $\cos(p\theta+\phi)$ or $\sin(p\theta+\phi)$ where $p \in \mathbb{Z}$ and ϕ is a constant phase term.

$$h_z = \left\{ \begin{array}{c} g(r) \\ h(r) \end{array} \right\} \cos(p\theta+\phi) \text{ or } \; h_z = \left\{ \begin{array}{c} g(r) \\ h(r) \end{array} \right\} \sin(p\theta+\phi) \tag{72}$$

where $g(r)$ is used to describe the field in the core region and $h(r)$ in the cladding region. Note that in both cases such dependence yield a p^2 in the propagation equation:

$$\frac{\partial^2 h_z}{\partial r^2} + \frac{1}{r}\frac{\partial h_z}{\partial r} + \left(k^2n(r)^2 - \beta^2 - \frac{p^2}{r^2}\right) h_z = 0 \tag{73}$$

The boundary condition requires that $g(a) = h(a)$ and using the cosine dependence on the magnetic field in eq. (72) in the eq. (73) yields

$$\frac{i\beta}{k^2n_{cl}^2-\beta^2}\frac{p}{a}h(a)\sin(p\theta+\phi) = \frac{i\beta}{k^2n_{co}^2-\beta^2}\frac{p}{a}g(a)\sin(p\theta+\phi) \tag{74}$$

which only holds if $p = 0$, which means that there is actually no dependence in θ! Since e_r and h_θ only depends on $\partial_\theta h_z$ there are both null: $e_r = 0$ and $h_\theta = 0$.
As we see the original is considerably simplified:

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \left(k^2n(r)^2 - \beta^2\right) h_z = 0 \tag{75a}$$

$$e_\theta = \frac{i\omega\mu_0}{k^2n^2-\beta^2}\frac{dh_z}{dr} \tag{75b}$$

$$h_r = \frac{-i\beta}{k^2n^2-\beta^2}\frac{dh_z}{dr} \tag{75c}$$

$$e_r = h_r = 0 \tag{75d}$$

In the core we have to solve

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \alpha^2 h_z = 0 \quad \text{with } \alpha^2 = k^2n_{co}^2 - \beta^2 \tag{76}$$

for with the solution[3] is the Bessel function of 0^{th}-order $J_0(\alpha r)$.

[3] Mathematically the 0^{th}-order Neumann function $N_0(\alpha r)$ is also a solution of this differential equation but it diverges at $r = 0$ and therefore cannot physical be suitable.

In the core we have to solve

$$\frac{d^2h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} - \gamma^2 h_z = 0 \qquad \text{with } \gamma^2 = \beta^2 - k^2 n_{cl}^2 \tag{77}$$

for with the solution[4] is the modified Bessel function of 1^{st}-order $I_0(\gamma r)$.

In conclusion we have the longitudinal component of the magnetic field inside the core and the cladding region:

$$h_z = \begin{cases} AJ_0(\alpha r) & \forall\, |r| \leq a \\ \\ BK_0(\gamma r) & \forall r > a \end{cases} \tag{78}$$

where A and B are two constant to be determined. The continuity of the field components h_z and e_θ at the boundary $r = a$ lead to

$$AJ_0(\alpha a) = BK_0(\gamma a) \tag{79a}$$

$$\frac{A}{\alpha} J_0'(\alpha a) = -\frac{B}{\gamma} K_0'(\gamma a) \tag{79b}$$

$$\Rightarrow \frac{J_0'(\alpha a)}{\alpha J_0(\alpha a)} = -\frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} \quad \Leftrightarrow \quad \frac{J_0'(U)}{UJ_0(U)} = -\frac{K_0'(W)}{WK_0(W)} \tag{80}$$

where we use the parameters $U = \alpha a$ and $W = \gamma a$ that are similar to the ones we introduced for the planar waveguide. Solving this eigenvalue problem yields the propagation constant β. Note that we can also use the relationship

$$J_0'(x) = -J_1(x) \tag{81a}$$

$$K_0'(x) = -K_1(x) \tag{81b}$$

Then we can rewrite the eigenvalue problem as

$$\boxed{\frac{J_1(U)}{UJ_0(U)} = -\frac{K_1(W)}{WK_0(W)}} \tag{82}$$

Strategy: The strategy[5] to obtain the modes in the fibre is the following:

1. solve the eigenvalue problem for a given set of parameter $\{a, n_{co}, n_{cl}\}$
2. Use the value of β to express the longitudinal components of the fields e_z and h_z
3. Use the longitudinal components to calculate the transverse components of the field. In the particular case of TE mode there are e_θ and h_r.

[4]Mathematically the modified Bessel function of 2^{nd} kind $K_0(\alpha r)$ is also a solution of this differential equation but it diverges at $r \to \infty$ and therefore cannot physical be suitable.

[5]Obviously this strategy is general and does not only apply to the TE mode.

In the particular case of TE mode the field in the core and cladding are

$$\text{in the core:} \begin{cases} e_\theta &= -i\omega\mu_0 \dfrac{a}{U} A J_1\left(\frac{U}{a}r\right) \\ h_r &= i\beta \dfrac{a}{U} A J_1\left(\dfrac{U}{a}r\right) \\ h_z &= A J_0\left(\dfrac{U}{a}r\right) \end{cases} \tag{83a}$$

$$\text{in the cladding:} \begin{cases} e_\theta &= i\omega\mu_0 \dfrac{a}{W} A K_1\left(\dfrac{W}{a}r\right) \\ h_r &= -i\beta \dfrac{a}{W} \dfrac{J_0(U)}{K_0(W)} A K_1\left(\dfrac{U}{a}r\right) \\ h_z &= \dfrac{J_0(U)}{K_0(W)} A K_0\left(\dfrac{W}{a}r\right) \end{cases} \tag{83b}$$

Note that the constant B in eq. (78) is replaced by $({}^{J_0(U)}/_{K_0(W)}\ A)$ calculated from boundary conditions at the interface core – cladding. To evaluate the constant A we use the **z**-component of the Poynting vector

$$S_z = \frac{1}{2}\left(\mathbf{E} \times \mathbf{H}^*\right) \cdot \hat{\mathbf{z}} = \frac{1}{2}\left(e_r H_\theta^* - e_\theta h_r^*\right) \tag{84}$$

and evaluate the optical power

$$P = \iint S_z \cdot r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} \left(e_r H_\theta^* - e_\theta h_r^*\right) \cdot r dr d\theta \tag{85}$$

TM modes This is the very same idea expect that this time $h_z = 0$. This yields that $e_\theta = h_r = h_z = 0$. In that case the eigenvalue equation is

$$\boxed{\frac{J_1(U)}{U J_0(U)} = -\left(\frac{n_{cl}}{n_{co}}\right)^2 \frac{K_1(W)}{W K_0(W)}} \tag{86}$$

and various components of the electromagnetic field

$$\text{in the core:} \begin{cases} e_r = i\beta \dfrac{a}{U} A J_1\left(\dfrac{U}{a}r\right) \\ e_z = A J_0\left(\dfrac{U}{a}r\right) \\ h_\theta = i\omega\epsilon_0 n_{co}^2 A\ J_1\left(\dfrac{U}{a}r\right) \end{cases} \tag{87a}$$

$$\text{in the cladding:} \begin{cases} e_r = -i\beta \dfrac{a}{W}\dfrac{J_0(U)}{K_0(W)} A K_1\left(\dfrac{W}{a}r\right) \\ e_z = \dfrac{J_0(U)}{K_0(W)} A K_0\left(\dfrac{W}{a}r\right) \\ h_\theta = -i\omega\epsilon_0 n_{cl}^2 \dfrac{a}{W}\dfrac{J_0(U)}{K_0(W)}\ K_1\left(\dfrac{W}{a}r\right) \end{cases} \tag{87b}$$

Hybrid modes In the case when neither e_z nor h_z is null[6] the solutions of the propagation equation eq. (64a) and (64b) will be given as a product of the n^{th}-order Bessel function and the azimuthal dependence $\cos(n\theta+\phi)$ or $\sin(n\theta+\phi)$ where $n \in \mathbb{Z}$. Since now both propagation equations are coupled we cannot simply eliminate the angular dependence ($n \neq 0$). Note however that the azimuthal dependence has to remain when crossing the interface core–cladding. Moreover in the expression of e_r, e_θ, h_r and h_θ appear terms as $\partial_r e_z$, $\partial_\theta e_z$ ... the azimuthal dependencies for the electric and the magnetic fields must follow

$$e_z = \begin{cases} A J_n\left(\dfrac{U}{a}r\right)\cos(n\theta+\phi) & \text{in the core} \\ A\dfrac{J_n(U)}{K_n(W)} K_n\left(\dfrac{W}{a}r\right)\cos(n\theta+\phi) & \text{in the core} \end{cases} \tag{88a}$$

$$h_z = \begin{cases} C J_n\left(\dfrac{U}{a}r\right)\sin(n\theta+\phi) & \text{in the core} \\ C\dfrac{J_n(U)}{K_n(W)} K_n\left(\dfrac{W}{a}r\right)\sin(n\theta+\phi) & \text{in the core} \end{cases} \tag{88b}$$

which we can use to express the e_r, e_θ and h_r, h_θ:

[6]The continuity conditions still need to hold!

- in the core ($|r| \leq a$):

$$e_r = \frac{-ia^2}{U^2}\left[A\beta\frac{U}{a}J_n'\left(\frac{U}{a}r\right) + C\omega\mu_0\frac{n}{r}J_n\left(\frac{U}{a}r\right)\right]\cos(n\theta+\phi) \tag{89a}$$

$$e_\theta = \frac{-ia^2}{U^2}\left[-A\beta\frac{n}{r}J_n\left(\frac{U}{a}r\right) - C\omega\mu_0\frac{u}{a}J_n'\left(\frac{U}{a}r\right)\right]\sin(n\theta+\phi) \tag{89b}$$

$$h_r = \frac{-ia^2}{U^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{n}{r}J_n\left(\frac{U}{a}r\right) + C\beta\frac{U}{a}J_n'\left(\frac{U}{a}r\right)\right]\sin(n\theta+\phi) \tag{89c}$$

$$h_\theta = \frac{-ia^2}{U^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{n}{r}J_n'\left(\frac{U}{a}r\right) + C\beta\frac{U}{a}J_n\left(\frac{U}{a}r\right)\right]\cos(n\theta+\phi) \tag{89d}$$

- in the cladding ($r > a$):

$$e_r = \frac{ia^2}{W^2}\left[A\beta\frac{W}{a}K_n'\left(\frac{W}{a}r\right) + C\omega\mu_0\frac{n}{r}K_n\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\cos(n\theta+\phi) \tag{90a}$$

$$e_\theta = \frac{ia^2}{W^2}\left[-A\beta\frac{n}{r}K_n\left(\frac{W}{a}r\right) - C\omega\mu_0\frac{W}{a}K_n'\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\sin(n\theta+\phi) \tag{90b}$$

$$h_r = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{co}^2\frac{n}{r}K_n\left(\frac{W}{a}r\right) + C\beta\frac{W}{a}K_n'\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\sin(n\theta+\phi) \tag{90c}$$

$$h_\theta = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{co}^2\frac{n}{r}K_n'\left(\frac{W}{a}r\right) + C\beta\frac{n}{r}K_n\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\cos(n\theta+\phi) \tag{90d}$$

Additionally we need to have continuity of the e_θ and h_θ at $r = a$, which leads to

$$A\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n = -C\omega\mu_0\left[\frac{J_n'(U)}{UJ_n(U)} + \frac{K_n'(W)}{WK_n(U)}\right] \tag{91a}$$

$$A\omega\epsilon_0\left[n_{cl}^2\frac{J_n'(U)}{UJ_n(U)} + n_{co}^2\frac{K_n'(W)}{WK_n(U)}\right] = -C\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n \tag{91b}$$

which has a solution for $\{A, C\}$ if and only if

$$\boxed{\begin{aligned}\left[\frac{J_n'(U)}{UJ_n(U)} + \frac{K_n'(W)}{WK_n(U)}\right]\left[n_{cl}^2\frac{J_n'(U)}{UJ_n(U)} + n_{co}^2\frac{K_n'(W)}{WK_n(U)}\right]\\ = \frac{\beta^2}{k^2}\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n^2 = \left(\frac{n_{co}^2}{U^2}+\frac{n_{cl}^2}{W^2}\right)n^2\end{aligned}} \tag{92}$$

which is the eigenvalue equation that we need to solve in order to find the propagation constant β. In this equation n is a positive integer number. As previously the equation must be solve numerically for a given value of the normalized frequency V.

On Fig. 8, we plot the eigenvalue equation for a 4μm-core diameter rod, which can be seen as an optical fibre with a silica core and an air-cladding. In that case the *normalized parametre* is $V = 12.395$. As we can see, this curve has several zeros, each of which corresponds to one particular value of U and therefore to one propagation constant β. As a general form the modes are labelled as HE_{nm} and EH_{nm} where n is the integer number used in eq. (92), corresponding to the order of the Bessel function used, and m is the index of the considered zero. Note though the the modes are alternating: the first one is HE_{11}. Then comes an EH mode and so on.

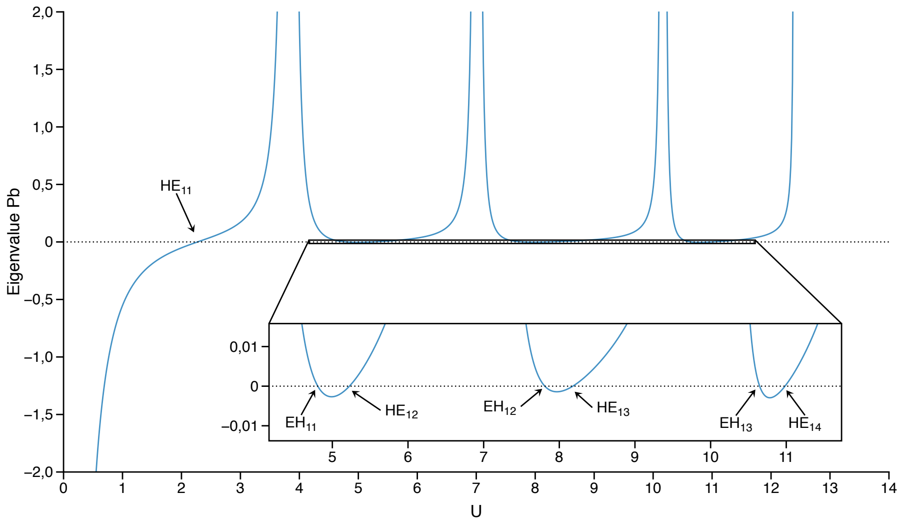

Figure 8: Eigenvalue problem eq. (92) for a fibre with a diameter $2a = 4\mu$m. The wavelength is 1064 nm. This corresponds to $V = 12.395$ for the considered waveguide (silica core surrounding by air). The arrow indicates the zeros and the corresponding optical modes.