in rectangular coordinates we would have:

$$G(f_x, f_y) = \iint_{-\infty}^{+\infty} g(x,y) \exp[-i2\pi(f_x x + f_y y)] \, dx\, dy$$

and we can simply use the fact that:

$$r = \sqrt{x^2+y^2} \quad \begin{matrix} \nearrow x = r\cos\theta \\ \searrow y = r\sin\theta \end{matrix} \qquad \theta = \arctg(y/x)$$

and in the optical frequency domain

$$\rho = \sqrt{f_x^2 + f_y^2} \quad \begin{matrix} \nearrow f_x = \rho\cos\phi \\ \searrow f_y = \rho\sin\phi \end{matrix} \qquad \phi = \arctg\left(\frac{f_y}{f_x}\right)$$

As a general form:
$$\mathcal{F}(g) = \hat{G}_0(\rho,\phi) = \int_0^{2\pi} \int_0^{+\infty} g_r(r) \exp[-i2\pi r\rho \underbrace{(\cos\theta\cos\phi + \sin\theta\sin\phi)}_{\cos(\theta-\phi)}] \, r\, dr\, d\theta$$

$$= \int_0^{\infty} \underbrace{\int_0^{2\pi} \exp[-i2\pi\rho . r \cos(\theta-\phi)]}\, g_r(r)\, r\, dr\, d\theta$$

Bessel function of 1st kind (zero order) $\Rightarrow J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-i a \cos(\theta-\phi)]\, d\theta$

$\Rightarrow G_0(\rho,\phi) = 2\pi \int_0^{\infty} g_R(r)\, J_0(2\pi r\rho) . r\, dr = \mathcal{B}(g(r))$ the Fourier transform is also circularly symmetric (no ϕ dependence!)

this is called Fourier-Bessel transform or Hankel transform of zero order.

inverse Fourier-Bessel transform: $g_R(r) = 2\pi \int_0^{\infty} G_0(\rho) . J_0(2\pi r\rho)\, \rho\, d\rho = \mathcal{B}^{-1}[G_0(\rho)]$

In rectangular coordinates we would have:

$$\hat{G}_{x}(f_x, f_y) = \iint_{-\infty}^{+\infty} g(x,y) \exp[-i2\pi(f_x x + f_y y)]\, dx\, dy$$

we can transform this Fourier transform by using the fact that

$$\begin{cases} r^2 = x^2+y^2 & // \; x = r\cos\theta & // \; y = r\sin\theta & // \; \theta = \operatorname{arctg}(y/x) \\ \rho^2 = f_x^2 + f_y^2 & f_x = \rho\cos\phi & f_y = \rho\sin\phi & \phi = \operatorname{arctg}(f_y/f_x) \end{cases}$$

$$\Rightarrow \mathcal{F}(g) = G_0(\rho, \phi) = \int_0^{2\pi}\int_0^{+\infty} g_r(r) \exp[-i2\pi\, r\rho\, \underbrace{(\cos\theta\cos\phi + \sin\theta\sin\phi)}_{\frac{1}{2}\cos(\theta-\phi)}]\, r\, dr\, d\theta$$

$$= \int_0^{\infty}\int_0^{2\pi} \exp[-i2\pi\rho r \cos(\theta-\phi)]\; g(r)\, r\, dr\, d\theta$$

And we can use the fact that the zero order Bessel function of 1st kind can be defined as

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-i\, a\cos(\theta-\phi)]\, d\theta$$

$$\Rightarrow G_0(\rho, \phi) = 2\pi\int_0^{+\infty} g_R(r)\, J_0(2\pi r\rho)\cdot r\, dr = \mathcal{B}[g_R(r)]$$ is the Fourier-Bessel transform

Note that as $g_r(r)$, it is circularly symmetric!

The inverse Fourier transform of a circularly symmetric spectrum $G_0(\rho)$ is

$$g_R(r) = 2\pi\int_0^{+\infty} G_0(\rho)\, J_0(2\pi r\rho)\, \rho\, d\rho$$

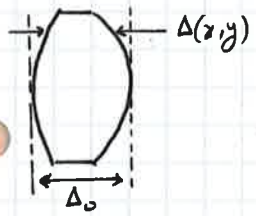
And due to similarity theorem we have: $\mathcal{B}[g_R(ar)] = \frac{1}{a^2}\, G_0\left(\frac{\rho}{a}\right)$

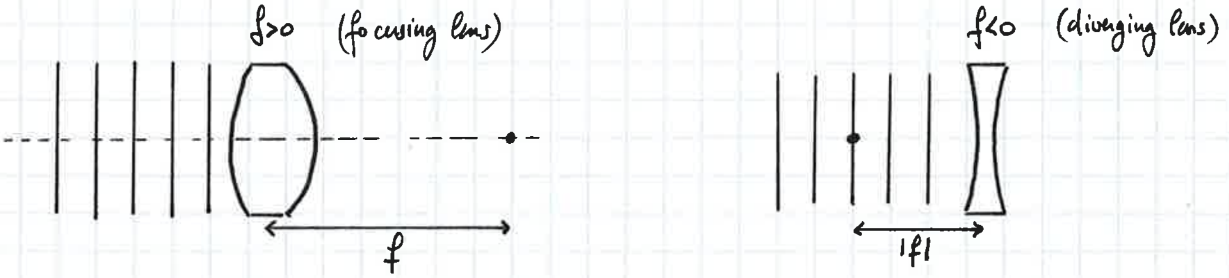
Fourier transforming Properties of lenses

We've already seen that the action of a lens can be described by the following transformation:

$$t_\ell(x,y) = \exp(ikn\Delta_0)\exp\left[ik(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] = \exp(ikn.\Delta_0)\exp\left[\frac{-ik}{2f}(x^2+y^2)\right]$$

focal of the lens

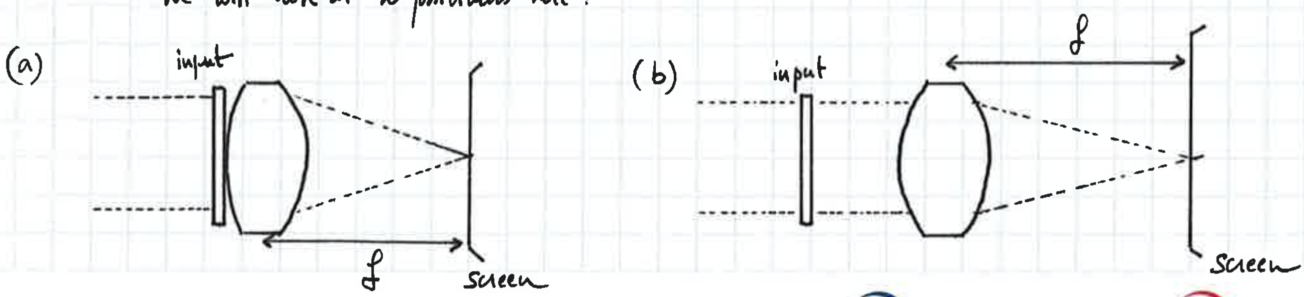




with $f>0$ the incident plane wave is transformed into spherical wave converging toward a point on the lens axis at a distance f behind the lens: the focal point.

Actually the converging lens has the inherent property to perform two-dimensional Fourier transform! This is a result of basic laws of propagation and diffraction of light. Here we assume the illumination to be monochromatic. The system can then be considered as "coherent": they are linear in complex amplitude.

We will look at 2 possibilities here:



Note that the input can be a photographic transparency or a more sophisticated device such as a spatial light modulator, where the amplitude transmittance is in response to an external applied electrical or optical information.

Case (a) input directly placed against the lens.

Since the input has its own amplitude transmittance $t_A(x,y)$ the field that arrives on the lens is:

$$U_l(x,y) = A \cdot t_A(x,y)$$

monochromatic plane wave with amplitude A.

Therefore after the lens:

$$U_l'(x,y) = U_l(x,y) \cdot P(x,y)\, e^{-\frac{ik}{2f}(x^2+y^2)}$$

this takes into account the fact that the lens is finite and acts like an aperture: $\begin{cases} P(x,y)=1 & \text{inside the lens/aperture} \\ P(x,y)=0 & \text{otherwise} \end{cases}$

To find the distribution on the screen (in the back focal plane) of the lens) we use the Fresnel diff. formula.

Reminder: $U(P) = \frac{-i\cos\delta}{\lambda} A \frac{e^{ik(r'+s')}}{r's'} \iint_A e^{ikf(\xi,\eta)}\, d\xi\, d\eta$ plane wave incident ($r' \to +\infty$)

with $f(\xi,\eta) = (l_0 - l)\xi + (m_0 - m)\eta + \left[\frac{1}{2}\left(\frac{1}{r'} + \frac{1}{s'}\right)(\xi^2+\eta^2) + \ldots\right]$ and $l - l_0 = \frac{x}{s'} + \frac{x_0}{r'} = x/z$

$m - m_0 = \frac{y}{s'} + \frac{y_0}{r'} = y/z$

$$\Leftrightarrow \quad U(P) = C \iint_{-\infty}^{+\infty} U(\xi,\eta)\, e^{-\frac{ik}{z}(x\xi + y\eta)}\, e^{\frac{ik}{2z}(\xi^2+\eta^2)}\, d\xi\, d\eta$$

In this integral $U(\xi,\eta) = U_\ell(\xi,\eta) \cdot P(\xi,\eta)\, e^{-\frac{ik}{2f}(\xi^2+\eta^2)}$ and $z = f$.

Therefore:

$$U(P) = c \iint_{-\infty}^{+\infty} U_\ell(\xi,\eta) \cdot P(\xi,\eta)\, \cancel{e^{-\frac{ik}{2f}(\xi^2+\eta^2)}}\, e^{-\frac{ik}{f}(x\xi+y\eta)}\, \cancel{e^{\frac{ik}{2f}(\xi^2+\eta^2)}}\, d\xi d\eta$$

and if the beam is smaller than the lens then ~~[illegible]~~ $P(\xi,\eta) = 1$, therefore:

$$U(P) = C \iint_{-\infty}^{+\infty} U_\ell(\xi,\eta)\, e^{-\frac{i2\pi}{\lambda f}(\xi x+\eta y)}\, d\xi d\eta$$

$\Rightarrow$ Looking in the focal plane of the lens gives the Fraunhofer diffraction pattern of the field incident on the lens.

Note that:
- $f_x = \frac{x}{\lambda f}$; $f_y = \frac{y}{\lambda f}$ are spatial frequencies.
- Measurement of the intensity distribution yields knowledge of the power-spectrum or more accurately the energy spectrum of the input:

$$I(u,v) = C^2 \left| \iint_{-\infty}^{+\infty} t_A(x,y)\, e^{-\frac{i2\pi}{\lambda f}(x.u+y v)}\, dx dy \right|^2$$

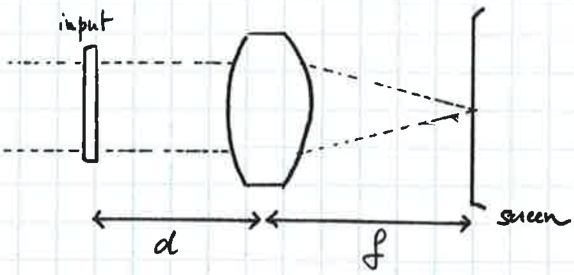
Note that we could have used the other expression of the Fresnel integral (see exercises)

$$U(x,y) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} U(\xi,\eta) \exp\left\{\frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\} d\xi d\eta$$

$$U(x,y) = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}(x^2+y^2)} \iint_{-\infty}^{+\infty} \left[U(\xi,\eta)\, e^{\frac{ik}{2z}(\xi^2+\eta^2)}\right] e^{-\frac{i2\pi}{\lambda z}(x\xi+y\eta)}\, d\xi d\eta$$

in order to see the phase term ... but since we can only access the intensity this doesn't make much sense!

case (b). Input placed in front of the lens (more general geometry!)



The input, located at a distance d from the lens is illuminated by a normally incident plane wave of amplitude A

Let's call $F_0(f_x, f_y)$ the Fourier spectrum of the light transmitted by the input transparency (t_A) and $F_\ell(f_x, f_y)$ the Fourier spectrum of the light incident on the lens:

$$F_0(f_x, f_y) = \mathcal{F}(At_A)$$

$$F_\ell(f_x, f_y) = \mathcal{F}(U_\ell)$$

$$F_\ell(f_x, f_y) = F_0(f_x, f_y) \cdot \exp\left[-i\pi\lambda . d\,(f_x^2 + f_y^2)\right]$$ free space propagation over d.

Assuming that the beam is smaller than the lens ($\forall (x,y)$, $P(x,y) = 1$)

$$U_f(u,v) = \frac{1}{i\lambda f} \exp\left[i\frac{k}{2f}(u^2+v^2)\right] \cdot \iint_{-\infty}^{+\infty} U_\ell(x,y) \exp\left[-i\frac{2\pi}{\lambda f}(xu + yv)\right] dx\,dy$$ (Goodman)

or (B&W) $$U(P) = C \iint_{-\infty}^{+\infty} U_\ell(x,y) \exp\left[-\frac{ik}{f}(x\xi + y\eta)\right] dx\,dy = C \iint_{-\infty}^{+\infty} U_\ell(x,y) e^{-\frac{i2\pi}{\lambda f}(x\xi + y\eta)} dx\,dy$$

we take the one with the phase term (Goodman.)

check this!

$$\Rightarrow U_f(u,v) = \frac{e^{i\frac{k}{2f}(u^2+v^2)}}{i\lambda f} \mathcal{F}(U_\ell) = \frac{e^{i\frac{k}{2f}(u^2+v^2)}}{i\lambda f} F_\ell\Big(\overbrace{\frac{u}{\lambda f}}^{f_x}, \overbrace{\frac{v}{\lambda f}}^{f_y}\Big)$$

and we know what F_ℓ is $F_\ell(f_x, f_y)$, therefore:

$$U_f(u,v) = \frac{e^{i\frac{k}{2f}(u^2+v^2)}}{i\lambda f} \times e^{-i\pi\lambda d\,\underbrace{(f_x^2+f_y^2)}_{\frac{u^2}{\lambda^2 f^2} + \frac{v^2}{\lambda^2 f^2}}} F_0(f_x, f_y)$$

$$= \frac{e^{\frac{ik}{2f}\left(1-\frac{d}{f}\right)(u^2+v^2)}}{i\lambda f} F_0\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right)$$

$$\Rightarrow \quad U_f(u,v) = \frac{A}{i\lambda f} \exp\left[i\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+v^2)\right] \times \iint_{-\infty}^{+\infty} t_A(\xi,\eta) \exp\left[-i\frac{2\pi}{\lambda f}(\xi u+\eta v)\right] d\xi\, d\eta$$

Amplitude and phase of the light at coordinate (u,v) are <u>again</u> related to the amplitude and phase of the input spectrum at frequencies $(u/\lambda f,\ v/\lambda f)$.

Note the quadratic phase factor in front of the integral! This term totally vanishes if $d=f$ leaving <u>exactly</u> the Fourier transform of the input