FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Part II Fourier optics.

Since 1930, optics has gradually developed closely with communication and information of electrical engineering. Both communication and imaging systems are designed to collect or convey information. For communication the information is generally of a temporal nature (modulation voltage or current waveform), while for imaging it is of a spatial nature (light amplitude or intensity distribution over space).

Strongest link is probably coming from mathematical tools that both domain share: Fourier analysis. But actually they also share properties: linearity and invariance. And they usually require description using frequency analysis. Just as an audio amplifier can easily be described in term of its (temporal) frequency response, it is often convenient to describe an imaging system in term of its (spatial) frequency response.

The history of optics is rich with example of important advances achieved by application of Fourier analysis: the Zernike phase-contrast microscope is an example that was worthy of a Nobel prize.

Fourier analysis in 2 dimensions

The Fourier transform (or Fourier spectrum or frequency spectrum) of a complex-valued function of 2 independent variables x & y is represented by

$$TF(g) = \hat{g} = \iint_{-\infty}^{+\infty} g(x,y)\, \exp[-i2\pi(f_x x + f_y y)]\, dx\, dy$$

f_x and f_y are then defined as "frequencies"

Similarly, the inverse Fourier transform of $\hat{G}(f_x; f_y)$ is given by

$$TF^{-1}(\hat{G}) = g(x,y) = \iint_{-\infty}^{+\infty} G(f_x, f_y)\, \exp[i2\pi(f_x x + f_y y)]\, df_x\, df_y$$

Note that these definitions are meaningful only if $g(x,y)$ has the following properties:

(1) g must be absolutely integrable over the infinite plane (x,y).

(2) g must have only a finite number of discontinuities and a finite nber of maxima and minima in any finite rectangle

(3) g must have no infinite discontinuities

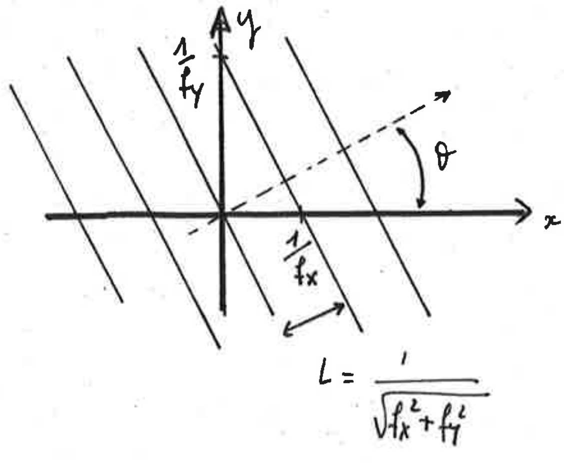
Meaning of the Fourier transform

Remember the familiar inverse Fourier transform: $g(t) = \int_{-\infty}^{+\infty} \hat{G}(f) \exp(i2\pi f \cdot t)\, df$

in this case, the time function $g(t)$ is expressed in term of its frequency spectrum, and it can be regarded as a decomposition of the function $g(t)$ into a linear combination of elementary functions $\exp(i2\pi f t)$. The complex number $\hat{G}(f)$ is simply a weighing factor that must be applied in order to synthetize $g(t)$.

In 2D it is similar: we have now a decomposition of the function $g(x,y)$ into a linear combination of elementary functions of the form $\exp[i2\pi(f_x x + f_y y)]$. Since these functions are 2π-periodic for any pair (f_x, f_y) has a phase that is $n \times 2\pi$ along lines described by

$$f_x x + f_y y = n \Rightarrow y = -\frac{f_x}{f_y} x + \frac{n}{f_y}$$



the elementary function may be regarded as being "directed" in the (x,y) plane at an angle θ (with respect to (Ox))

$$\theta = \operatorname{arctg}\left(\frac{f_y}{f_x}\right)$$

Basics theorems. These are very similar to one dimension statements and are mainly extension.

(i) linearity

$$\mathcal{F}(\alpha g + \beta h) = \alpha \mathcal{F}(g) + \beta \mathcal{F}(h)$$

(ii) Similarity

if $\mathcal{F}[g(x,y)] = \hat{G}(f_x, f_y)$ then $\mathcal{F}[g(ax, by)] = \frac{1}{|ab|} \hat{G}\left(\frac{f_x}{a}, \frac{f_y}{b}\right)$

a "stretch" of coordinate in the space domain results in a contraction in the spatial frequency domain and a change of amplitude.

(iii) shift theorem

if $\mathcal{F}[g(x,y)] = \hat{G}(f_x, f_y)$ then $\mathcal{F}[g(\underbrace{x-a, y-b}_{\text{translation}})] = G(f_x, f_y) \exp[\underbrace{-i2\pi(f_x a + f_y b)}_{\text{phase-shift.}}]$

(iv) Parseval's theorem

if $\mathcal{F}(g(x,y)) = \hat{G}(f_x, f_y)$ then $\underbrace{\iint_{-\infty}^{+\infty} |g(x,y)|^2 \, dx\, dy}_{\text{energy in } g(x,y)} = \underbrace{\iint_{-\infty}^{+\infty} |\hat{G}(f_x, f_y)|^2 df_x \, df_y}_{|\hat{G}(f_x,f_y)|^2 \text{: spectral density of energy}}$

(v) Convolution

if $\mathcal{F}[g(x,y)] = \hat{G}(f_x, f_y)$ and $\mathcal{F}[h(x,y)] = \hat{H}(f_x, f_y)$

then

$$\mathcal{F}\left\{ \iint_{-\infty}^{+\infty} g(\xi, \eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta \right\} = \hat{G}(f_x, f_y)\, H(f_x, f_y)$$

the Fourier transform of a convolution of 2 functions is simply the product of each Fourier transform.

Separable functions

A function $g(x,y)$ is called separable if we can write is as [redacted]

in rectangular coordinates : $g(x,y) = g_X(x)\, g_Y(y)$

polar coordinates $g(r,\theta) = g_R(r)\, g_\Theta(\theta)$

these functions have interesting properties regarding Fourier transforms:

(i) rectangular coordinates.

$$\mathcal{F}[g(x,y)] = \iint g(x,y) \exp[-i2\pi(f_x x + f_y y)]\, dx\, dy = \int_{-\infty}^{+\infty} g_X(x) \exp(-i2\pi f_x x)\, dx \int_{-\infty}^{+\infty} g_Y(y) \exp(-i2\pi f_y y)\, dy$$

$$\boxed{\mathcal{F}[g(x,y)] = \mathcal{F}_X(g_X)\, \mathcal{F}_Y(g_Y)}$$

(ii) polar coordinates

it can be shown that

$$\mathcal{F}[g(r,\theta)] = \sum_{k=-\infty}^{+\infty} c_k (-i)^k e^{ik\phi} \underbrace{\mathcal{H}_k}\{g_R(r)\}$$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} g_\Theta(\theta)\, e^{-ik\theta}\, d\theta$$

Hankel transform operator of order k defined as

$$\mathcal{H}_k[g_R(r)] = 2\pi \int_0^{+\infty} r\, g_R(r)\, \underbrace{J_k}(2\pi r\rho)\, dr$$

k-th order Bessel function of 1st kind.

(iii) functions with circular symmetry

functions with circular symmetry are the simplest case of functions separable in polar coordinates. They can be expressed as

$$g(r,\theta) = g_R(r)$$