Basics of Lasers

Laser cavities

3.1 Introduction

The first role of the laser cavity is to provide a positive feedback in order to enhance the gain[1]. Obviously, spectral properties of both the gain medium and the cavity will affect the final properties of the laser beam. In this chapter, we will look at the spectral properties of the cavity, but also see how the cavity can shape the spatial structure of the laser beam.
The cavity will affect

1. *Direction of emission.* The photons created by spontaneous emission will nearly all be eliminated by the cavity expect the one needed for the laser itself. Frequency response of the cavity itself will influence the operating wavelength of the laser.
2. *Directivity.* The phenomenon of diffraction in combination with the different elements, which constitute the laser cavity will lead to the formation of modes. These modes have a spatial structure, with a distribution of the amplitude and the phase.
3. Frequency response of the cavity itself will influence the operating wavelength of the laser.
4. *Polarization.* The presence of a Brewster-cut element will strongly enhance the loss of one of the polarization state of the generated beam.

In general, the more severe the selections, the higher the coherence properties of the beam, in term of directivity (spatial modes), degree of polarization and purity of the emitted frequency. By contrast with a real cavity, we will limit our study to empty resonators. Real cavity includes an active medium and its complete study is usually rather difficult. However experiments show that the general behavior obtained with empty resonator can be applied to real laser cavity. Note that nonlinear processes can happen due to the presence of the active medium such as competition between modes, frequency detuning, or thermal lens.

[1]Note that the cavity is particularly necessary in the case of low-gain. However there exists laser for which the gain is so high that a single pass can be enough. We then talk about *superradiance.* We can cite copper-vapor laser, N_2 laser working in the ultraviolet, and free-electron lasers.

3.2 Fabry-Pérot cavity

Let's consider a Fabry-Pérot cavity. It is made of two parallel interfaces. The coefficients for the reflected and transmitted field at the interfaces are respectively r_i and t_i. These coefficients are given for the amplitudes. We assume that the incident wave is plane, with a wavevector $\vec{k}$.

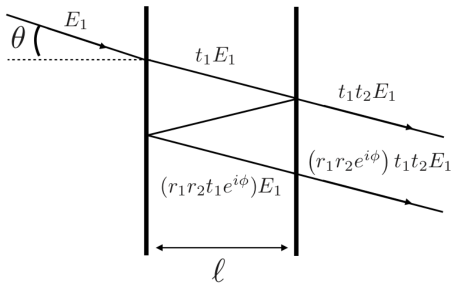

Figure 3.1: Schematic of a Fabry-Pérot cavity.

If the wave enters at normal incidence, then

$$\phi = 2\ell k = \frac{4\pi L\nu}{c} = \frac{2\pi\nu}{\Delta\nu_F} \tag{3.1}$$

where $\Delta\nu_F$ is the free spectral range for the cavity. The output field is the superposition

$$E_t = E_1t_1t_2\left[1 + r_1r_2e^{i\phi} + \left(r_1r_2e^{i\phi}\right)^2 + \dots\right] = E_1t_1t_2\frac{1-\left(r_1r_2e^{i\phi}\right)^n}{1-r_1r_2e^{i\phi}} \tag{3.2}$$

We can then write the power transmission of the Fabry-Pérot interferometer as $T_{\mathrm{FP}} = |E_t|^2/|E_0|^2$:

$$T_{\mathrm{FP}} = \frac{t_1^2t_2^2}{\left(1-r_1r_2e^{i\phi}\right)\left(1-r_1r_2e^{-i\phi}\right)} = \frac{t_1^2t_2^2}{1-2r_1r_2\cos\phi + \left(r_1r_2\right)^2} \tag{3.3}$$

Using the trigonometric identity $\cos 2\theta = 1 - 2\sin^2\theta$, we can then write this equation as

$$T_{\mathrm{FP}} = \frac{t_1^2t_2^2}{\left(1-r_1r_2\right)^2 + 4r_1r_2\sin^2\dfrac{\phi}{2}} \tag{3.4}$$

Since reflection and transmission coefficients are $R_i = r_i^2 = 1 - t_i^2 = 1 - T_i$, the eq. (3.4) becomes:

$$T_{\mathrm{FP}} = \frac{\left(1-R_1\right)\left(1-R_2\right)}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\sin^2\dfrac{\phi}{2}} \tag{3.5}$$

In the case of identical mirrors eq. (3.5) simply becomes

$$T_{\mathrm{FP}} = \frac{1}{1 + \dfrac{4R}{(1-R)^2}\sin^2\dfrac{\phi}{2}} = \frac{1}{1 + \dfrac{4R}{(1-R)^2}\sin^2\left(\dfrac{\pi\nu}{\Delta\nu_F}\right)} \tag{3.6}$$

Fig. 3.2 shows the transmission of a Fabry-Pérot interferometer for two different values of the coefficient of reflection (R). As the figure shows, the larger R, the sharper the resonance. This also corresponds to a cavity, with a much longer lifetime. There are two ways to characterize the cavity whether we only consider one resonance (fig. 3.2b) or the overall response (fig. 3.2a). In the first case, we are talking about *quality factor* by analogy with resonance phenomena that exist in other domain of physics. In the second case, we use the *finesse*.

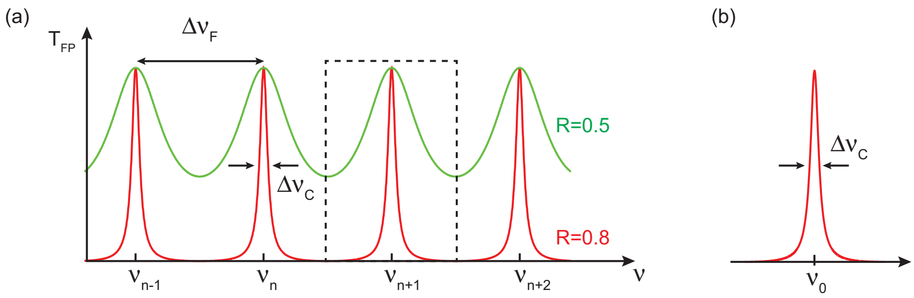

Figure 3.2: (a) Transmission of a Fabry-Pérot interferometer for $R = 0.8$ and $R = 0.5$ and (b) one isolated peak of the overall response function.

3.2.1 Quality factor and photon lifetime

definition

If we only consider one resonance peak of the frequency comb that describe the transmission of the Fabry-Pérot interferometer (fig. 3.2b), we can simply define the quality factor as

$$Q = \frac{\nu_0}{\Delta\nu_c} = 2\pi\,\frac{\text{Energy stored}}{\text{Energy lost per cycle}} \tag{3.7}$$

where ν_0 is the central frequency of the resonance. As we will see the bandwidth of the resonance $\Delta\nu_c$ is directly linked to the lifetime of the photons in the cavity τ_c and therefore Q can also be written as $Q = 2\pi\nu_0\tau_c$. For the present case, the energy is proportional to the number of photons and the energy lost to the change of photon number. Therefore[2]

$$Q = 2\pi \frac{h\nu\mathcal{J}}{h\nu\left(-\dfrac{d\mathcal{J}}{dt}\right)\times\dfrac{1}{\nu}} = -\frac{2\pi\nu\mathcal{J}}{\left(\dfrac{d\mathcal{J}}{dt}\right)} \tag{3.8}$$

This can be rewritten as

$$\frac{d\mathcal{J}}{dt} = -\frac{\omega}{Q}\mathcal{J} \;\Leftrightarrow\; \frac{d\mathcal{I}}{dt} = -\frac{\omega}{Q}\mathcal{I} \tag{3.9}$$

which is readily solved as

$$I(t) = I_0 e^{-\dfrac{\omega}{Q}t} = I_0 e^{-\left(\dfrac{t}{\tau_c}\right)} \tag{3.10}$$

This equation clearly shows the link between the lifetime of the photon in the cavity, and the quality factor. We remind that we defined previously the lifetime of the photon inside the cavity as $k = 1/\tau_c$.

case of the Fabry-Pérot cavity

In the case of a Fabry-Pérot cavity with R_1 and R_2 the reflection coefficient of the mirror (Fig. 3.3). The length of the cavity is ℓ. The loss per unit of length inside the cavity is α.

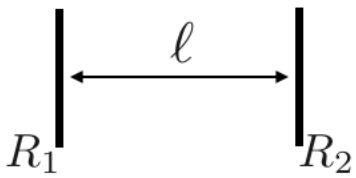

Figure 3.3: Fabry-Pérot resonator

Suppose that we have an initial intensity I_0 inside the cavity. After one round trip the intensity is then

$$I = I_0 R_1 R_2 (1-\alpha\ell)^2 \tag{3.11}$$

After p round-trip, this intensity is now

$$I = I_0\left[R_1 R_2 (1-\alpha\ell)^2\right]^p = I_0 \exp\left(-\frac{t}{\tau_c}\right) \tag{3.12}$$

[2]Note that the cycle is $1/\nu$, where ν is the frequency of the light. This is actually the only thing oscillating here (electric field) since there is only one peak, and so the cavity is totally absent in this description.

where p can be expressed by

$$p = \frac{(c/n)\, t}{2\ell} \tag{3.13}$$

From eq. (3.12) and (3.13), we get

$$\frac{1}{\tau_c} = -\left(\frac{c}{2n\ell}\right) \ln\left[R_1 R_2 (1 - \alpha\ell)^2\right] = \frac{\omega}{Q} \tag{3.14}$$

and therefore

$$Q = \frac{-2n\ell\omega}{c} \frac{1}{\ln\left[R_1 R_2 (1 - \alpha\ell)^2\right]} = \frac{\nu_0}{\Delta\nu_c} = 2\pi\nu\tau_c \tag{3.15}$$

order of magnitude

Let a 90 cm-long He-Ne laser ($\lambda = 633$ nm) composed of 2 mirrors with equal coefficient of reflection ($R = 0.98$) and no additional loss ($\alpha = 0$) and $n = 1$. Let's calculate the cavity lifetime τ_c as well as the quality factor.
From eq. (3.14), we have

$$\frac{1}{\tau_c} = -\frac{2 \ln R}{\left(\frac{2\ell}{c}\right)} \approx 1/150 \text{ ns}^{-1} \tag{3.16}$$

The optical frequency of the laser is $\nu_0 = c/\lambda = 5 \times 10^{14}$ Hz and therefore the quality factor $Q = 2\pi\nu_0\tau_c = 2\pi \times 5 \times 10^{14} \times 150$ ns $= 4.7 \times 10^8$.

3.2.2 Finesse

definition

For optical resonator, the Q-factor is usually large. It is then more convenient to define another factor: the *finesse*. It takes into account both the resonance bandwidth and the spacing between peaks and is define as

$$F = \frac{\Delta\nu_F}{\Delta\nu_c} \tag{3.17}$$

case of Fabry-Pérot

From eq. (3.5), we can calculate the maximum and minimum of the transmission function (fig. 3.4).

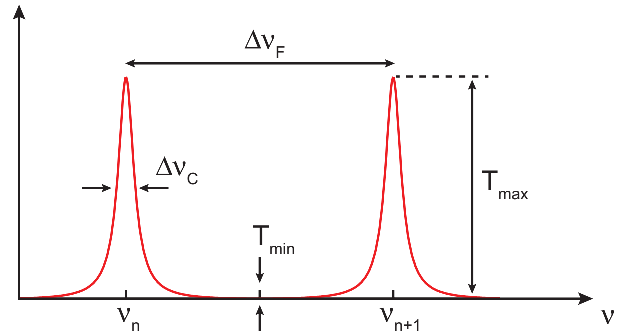

Figure 3.4: Transmission of a Fabry-Pérot interferometer.

These correspond respectively to $\sin^2\phi = 0$ or 1, and are readily given by

$$T_{\text{min}} = \frac{(1-R_1)(1-R_2)}{\left[1+\sqrt{R_1R_2}\right]^2} \tag{3.18a}$$

$$T_{\text{max}} = \frac{(1-R_1)(1-R_2)}{\left[1-\sqrt{R_1R_2}\right]^2} \tag{3.18b}$$

To calculate the width of the transmission peak, let's start from the transmission function (eq. (3.5))

$$T = \frac{T_1T_2}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\sin^2 \phi/2} \tag{3.19}$$

which we will write in the form

$$T = \frac{T_1T_2}{\left(1-\sqrt{R_1R_2}\right)^2} \underbrace{\frac{1}{1+\dfrac{4\sqrt{R_1R_2}\sin^2\phi/2}{\left(1-\sqrt{R_1R_2}\right)^2}}}_{\text{lorentzian}} \tag{3.20}$$

Looking only at the Lorentzian part, and noting that in the vicinity of a resonance peak at $\phi = \phi_{\text{res.}}$

$$\sin^2\frac{\phi}{2} \approx \frac{1}{4}\left(\phi-\phi_{\text{res.}}\right)^2 \tag{3.21}$$

we then write

$$\frac{1}{1+\dfrac{4\sqrt{R_1R_2}\sin^2\phi/2}{\left(1-\sqrt{R_1R_2}\right)^2}} = \frac{1}{1+\dfrac{\left(\phi-\phi_{\text{res.}}\right)^2}{\left(\dfrac{1-\sqrt{R_1R_2}}{\sqrt[4]{R_1R_2}}\right)^2}} \tag{3.22a}$$

The FWHM of this peak is then[3]

$$\Delta\phi = \frac{2\left(1-\sqrt{R_1R_2}\right)}{\sqrt[4]{R_1R_2}} = \frac{2\pi\Delta\nu_c}{\Delta\nu_F} \tag{3.23}$$

Since the finesse is defined as $F = {}^{\Delta\nu_F}/_{\Delta\nu_c}$ we deduce:

$$F = \frac{\pi\sqrt[4]{R_1R_2}}{1-\sqrt{R_1R_2}} = \frac{\pi\sqrt{R}}{1-R} \tag{3.24}$$

when $R_1 = R_2 = R$.

3.3 Stability of a resonator

3.3.1 General case

Consider the spherical-spherical cavity (fig. 3.5). where R_1 and R_2 are the radii

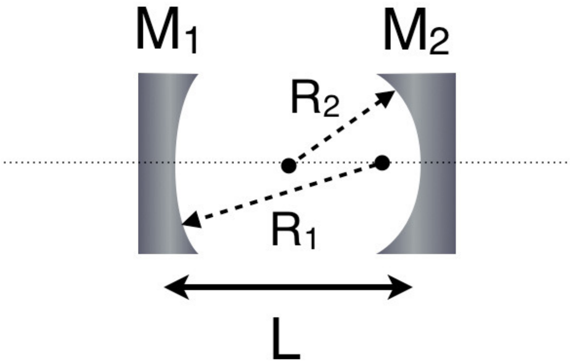

Figure 3.5: Spherical-Spherical resonator

of curvature of the mirrors. At this stage, the main question is to know whether such a cavity if optically stable or not. The cavity can be describes (one return) by the following ABCD matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

And as a consequence, wondering whether the cavity is stable is equivalent to questioning if an optical beam will remain inside the cavity or not. From the

[3]The FWHM of a curve $f(x) = \dfrac{1}{1+\left(\dfrac{x}{\sigma}\right)^2}$ is 2σ.

ABCD matrix of the cavity, we can evaluate the characteristics of the beam (ray optics) after one round (r_1, θ_1) from an initial beam (r_0, θ_0), and repeat this for N return inside the cavity. Note that such cavity is equivalent to the one presented on fig.

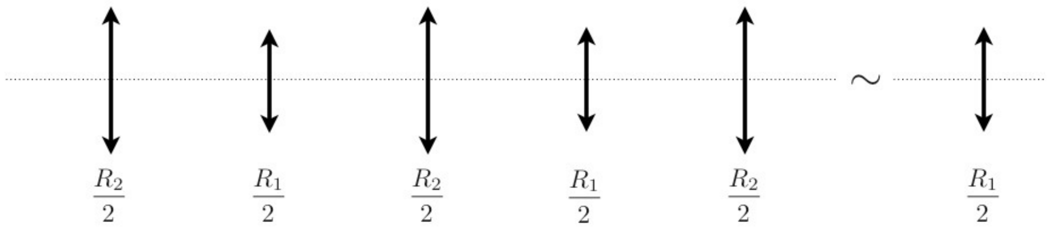

Figure 3.6: Expanded view of a spherical-spherical cavity

$$\begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} \rightarrow \begin{pmatrix} r_1 \\ \theta_1 \end{pmatrix} = M \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} \rightarrow \begin{pmatrix} r_2 \\ \theta_2 \end{pmatrix} = M \begin{pmatrix} r_1 \\ \theta_1 \end{pmatrix} = M \times M \times \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix}$$

After N round trip, this beam is then:

$$\begin{pmatrix} r_N \\ \theta_N \end{pmatrix} = M^N \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} \tag{3.25}$$

The cavity is stable if and only if

$$\lim_{N \to \infty} \begin{pmatrix} r_N \\ \theta_N \end{pmatrix} \text{remains finite}$$

In the present case, one way to figure out the stability of the problem is to look at its eigenvalue. Indeed, when λ_1, λ_2 the eigenvalues of M, then

$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Longrightarrow M^N = \begin{bmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{bmatrix}$$

From the eigenvalues, we can say that the system is stable if and only if both $|\lambda_1|$ and $|\lambda_2$ are both less that 1. To find the eigenvalue we need to solve $\det(M - \lambda \mathbb{I}) = 0$. Note that $\det(M) = 1$.

$$\begin{vmatrix} A - \lambda & B \\ C & D\lambda \end{vmatrix} = (A - \lambda)(D - \lambda) - CB = 0$$

$$\lambda^2 - (A + D)\lambda + 1 = 0 \tag{3.26}$$

From this equation, we can express the product and the sum of the solution (λ_1, λ_2):

$$\lambda_1 + \lambda_2 = A + D \tag{3.27a}$$

$$\lambda_1 \lambda_2 = 1 \tag{3.27b}$$

and (λ_1, λ_2) are:

$$\lambda_{1,2} = \left(\frac{A+D}{2}\right) \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} \tag{3.28}$$

- if $(A + D)^2 > 4$ then $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. However, because of eq. (3.27b), then one of the eigenvalue is larger than 1: the cavity is not stable.
- if $(A + D)^2 < 4$ then $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. The eigenvalues are then

  $$\lambda_{1,2} = \left(\frac{A+D}{2}\right) \pm i \sqrt{\left(\frac{A+D}{2}\right)^2 - 1}$$

  and $\mathrm{Re}(\lambda) < 1$: the cavity is stable

conclusion: the cavity is stable if and only if

$$\left|\frac{A+D}{2}\right| \leq 1 \tag{3.29}$$

3.3.2 Spherical-spherical cavity

Let calculate the matrix for the return in the spherical-spherical cavity (Fig. 3.7)

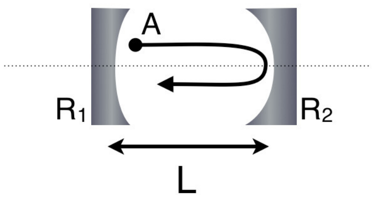

Figure 3.7: spherical-spherical cavity

$$M = \begin{bmatrix} 1 & L \\ -\frac{1}{f_2} & 1 - \frac{L}{f_2} \end{bmatrix} \begin{bmatrix} 1 & L \\ -\frac{1}{f_1} & 1 - \frac{L}{f_1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{L}{f_1} & - \\ - & 1 - \frac{2L}{f_2} - \frac{L}{f_1} + \frac{L^2}{f_1 f_2} \end{bmatrix} \tag{3.30}$$

Then we can write the stability condition:

$$\frac{1}{2}\ \mathrm{tr}(M) = 1 - \frac{L}{f_2} - \frac{L}{f_1} + \frac{L^2}{2f_1f_2} = 2\left(1 - \frac{L}{2f_1}\right)\ \left(1 - \frac{L}{2f_2}\right) - 1 \tag{3.31}$$

And since $R_i = 2f_i$ the condition of stability is:

$$0 \leq \underbrace{\left(1 - \frac{L}{R_1}\right)\ \left(1 - \frac{L}{R_2}\right)}_{g_1g_2} \leq 1 \tag{3.32}$$

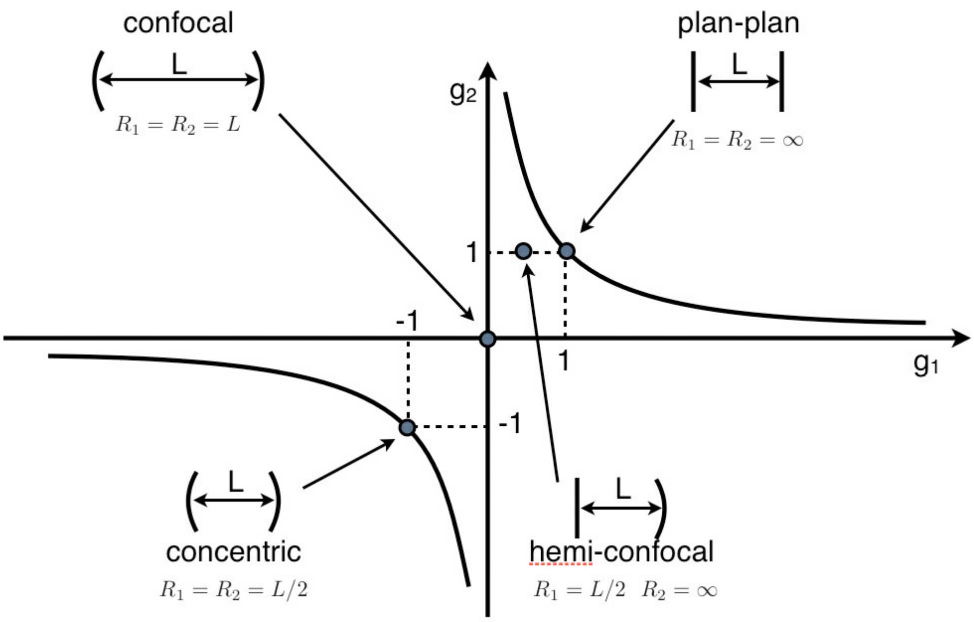

Figure 3.8: Graphical representation of the stability of a cavity. It is stable for $g_1g_2 \leq 1$.

3.3.3 Semi-spherical cavity

Evolution of the beam waist

Such cavity is composed by a flat mirror and a spherical one, with R its radius of curvature. ($R > 0$ for concave mirror). Distance between both mirrors is L. On

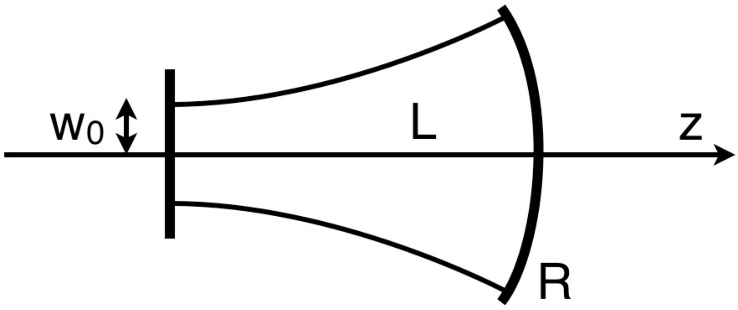

Figure 3.9: Semi-spherical cavity

the flat mirror we have:

$$w^2(L) = w_0^2\left[1+\left(\frac{\lambda L}{\pi w_0^2}\right)^2\right] \tag{3.33a}$$

$$R(L) = L\left[1+\left(\frac{\pi w_0^2}{\lambda L}\right)^2\right] \tag{3.33b}$$

Dividing eq. (3.33a) by eq. (3.33b) we obtain[4]:

$$\frac{\pi w^2}{\lambda R} = \frac{\lambda L}{\pi w_0^2} \tag{3.34}$$

Using eq. (3.34), we can isolate w^2 from in the eq. (3.33a) and write w and w_0 as a function of the geometrical parameters of the cavity:

$$w_0^4 = \left(\frac{\lambda}{\pi}\right)^2 L(R-L) \tag{3.35a}$$

$$w^4 = \left(\frac{\lambda}{\pi}\right)^2 \frac{R^2 L}{R-L} \tag{3.35b}$$

Evolution of the phase

After one round trip inside the cavity, the accumulated phase must be related to 2π by[5]

$$2kL - 2(m+n+1)\Phi = 2q\pi \tag{3.36}$$

[4] $w^2 = w_0^2(1+\alpha)$ and $R = L\left(1+{}^1\!/\!_\alpha\right)$ with $\alpha = \left({}^{\lambda L}\!/\!_{\pi w_0^2}\right)^2 \Rightarrow {}^{w^2}\!/\!_R = \left({}^{w_0^2}\!/\!_L\right)\alpha = \left({}^{w_0^2}\!/\!_L\right)\left({}^{\lambda L}\!/\!_{\pi w_0^2}\right)$

[5] Using the Laguerre-Gauss representation, the factor for the Gouy phase is $(2p+l+1)$.

where q is an integral multiple of half-wavelength. In this particular case the Gouy phase is simply given by[6]

$$\Phi = \mathrm{atan}\sqrt{\frac{\mathrm{L}}{\mathrm{R}-\mathrm{L}}} = \mathrm{acos}\sqrt{1-\frac{\mathrm{L}}{\mathrm{R}}} \tag{3.37}$$

The resonance frequencies are then given by

$$\nu_{mnq} = \Delta\nu_F \left[q + \frac{1}{\pi}(m+n+1)\,\mathrm{acos}\,\sqrt{1-\frac{\mathrm{L}}{\mathrm{R}}}\right] \tag{3.38}$$

3.3.4 General case of Fabry-Pérot cavity

We can consider any Fabry-Pérot cavity as two Semi-spherical cavity with z_1 and z_2 their respective length. Since the waist is the same for both Semi-spherical cavity, from eq. (3.35a), we have:

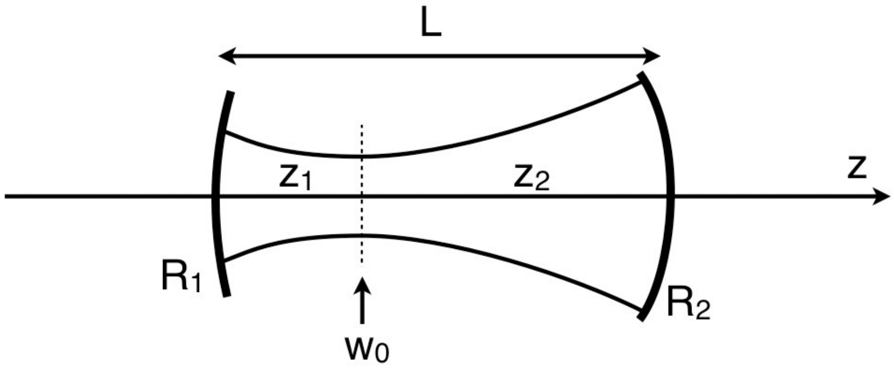

Figure 3.10: spherical-spherical resonator

$$z_2(R_2 - z_2) = z_1(R_1 - z_1) \tag{3.39}$$

and since $z_1 + z_2 = L$ we can express z_1 and z_2 with the parameter of the cavity L, R_1 and R_2:

$$z_1 = \frac{L(R_2 - L)}{R_1 + R_2 - 2L} \tag{3.40a}$$

$$z_2 = \frac{L(R_1 - L)}{R_1 + R_2 - 2L} \tag{3.40b}$$

[6]This uses the fact that $\mathrm{atan}\,\alpha = \mathrm{acos}\sqrt{\frac{1}{1+\alpha^2}}$.

Since the radius of curvature is determined by both mirror, from eq. (3.35b, 3.40a, 3.40b) we have:

$$w_1^4 = \left(\frac{\lambda R_1}{\pi}\right)^2 \frac{R_2 - L}{R_1 - L} \frac{L}{R_1 + R_2 - L} \tag{3.41a}$$

$$w_2^4 = \left(\frac{\lambda R_2}{\pi}\right)^2 \frac{R_1 - L}{R_2 - L} \frac{L}{R_1 + R_2 - L} \tag{3.41b}$$

When cavity is stable, we can express the waist w_0 by (we assume here that $L < R_1 + R_2$):

$$w_0^4 = \left(\frac{\lambda}{\pi}\right)^2 \frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2} \tag{3.42}$$

Resonance condition for a spherical-spherical resonator

We already saw that the resonance condition is given by[7] (Eq. (3.38)

$$\nu_{mnq} = \Delta\nu_F \left[q + \frac{1}{\pi}(m + n + 1)\,\Phi\right]$$

In this case we can separate the spherical-spherical cavity into two semi-spherical cavity in order to calculate the Gouy phase

$$\Phi = \mathrm{atan}\frac{\mathrm{z}_1}{\mathrm{Z_R}} + \mathrm{atan}\frac{\mathrm{z}_2}{\mathrm{Z_R}} = \mathrm{atan}\left(\frac{\lambda \mathrm{z}_1}{\pi \mathrm{w}_0^2}\right) + \mathrm{atan}\left(\frac{\lambda \mathrm{z}_2}{\pi \mathrm{w}_0^2}\right) \tag{3.43}$$

Using Eq. (3.40a), (3.40b), (3.42) we can write

$$\frac{\lambda z_1}{\pi w_0^2} = \sqrt{\frac{L\,(R_2 - L)}{(R_1 - L)\,(R_1 + R_2 - L)}} \tag{3.44}$$

$$\frac{\lambda z_2}{\pi w_0^2} = \sqrt{\frac{L\,(R_1 - L)}{(R_2 - L)\,(R_1 + R_2 - L)}} \tag{3.45}$$

which lead to[8]

$$\Phi = \mathrm{acos}\sqrt{\left(1 - \frac{\mathrm{L}}{\mathrm{R}_1}\right)\left(1 - \frac{\mathrm{L}}{\mathrm{R}_2}\right)} \tag{3.46}$$

[7] Using the Laguerre-Gauss representation, the factor for the Gouy phase is $(2p + l + 1)$.

[8] We remind the trigonometric relationship $\mathrm{atan\,a} + \mathrm{atan\,b} = \mathrm{acos}\sqrt{\frac{(1 - \mathrm{ab})^2}{1 + \mathrm{a}^2 + \mathrm{b}^2 + (\mathrm{ab})^2}}$

We can generalize the form of the Gouy phase for any stable resonator and write the resonance conditions as

$$\nu_{mnq} = \Delta_F \left[q + \frac{1}{\pi} (m + n + 1) \, \mathrm{acos} \pm \sqrt{\mathrm{g_1 g_2}} \right] \tag{3.47}$$

where the (+) sign (resp. (−) sign) corresponds to g_1 and g_2 positive (resp. negative).

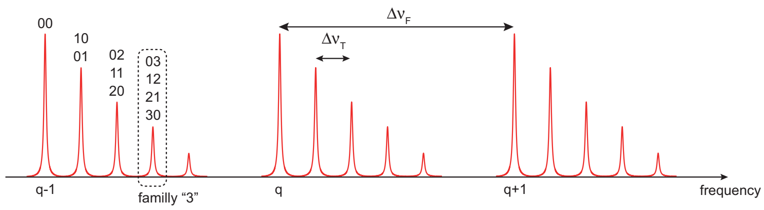

Figure 3.11: Resonance frequencies for Hermite-Gauss transverse mode in a resonator

3.3.5 confocal resonator

In the particular case of a confocal resonator, we have $R_1 = R_2 = L$ and then eq. (3.41a) becomes:

$$w = \sqrt{\frac{\lambda L}{\pi}} \tag{3.48}$$

3.4 ABCD law for a cavity

By definition of the q-parameter and after a return-trip inside the cavity we can write:

$$q(z + 2L) = \frac{A \; q(z) + B}{C \; q(z) + D} = q(z) \tag{3.49}$$

that we can rewrite as:

$$B \left(\frac{1}{q} \right)^2 + (A - D) \frac{1}{q} - C = 0 \tag{3.50}$$

for which roots are given by

$$\frac{1}{q} = -\frac{A - D}{2B} \pm \frac{\sqrt{(A - D)^2 + 4BC}}{2B} \tag{3.51}$$

Moreover, since $AD - BC = 1$ and the cavity is stable (then the discriminant of eq. (3.50) is negative) we have:

$$\frac{1}{q} = \frac{D-A}{2B} - \frac{i}{B}\sqrt{1-\left(\frac{A+D}{2}\right)^2} = \frac{1}{R} - \frac{i\lambda}{\pi w^2} \tag{3.52}$$

As a result we have:

$$R(z) = \frac{2B}{D-A} \tag{3.53a}$$

$$w^2(z) = \frac{\lambda B}{\pi}\frac{1}{\sqrt{1-\left(\frac{A+D}{2}\right)^2}} \tag{3.53b}$$

We can also express the Gouy phase over one round trip:

$$\Phi = \cos^{-1}\sqrt{\frac{A+D}{2}} = \cos^{-1}\sqrt{g_1 g_2} \tag{3.54}$$

which yields the resonance frequencies[9]

$$\nu_{mnq} = \Delta\nu_F\left[q + \frac{1}{\pi}(m+n+1)\,\mathrm{acos}\sqrt{\frac{\mathrm{A+D}}{2}}\right] \tag{3.55}$$

[9]Using the Laguerre-Gauss representation, the factor for the Gouy phase is $(2p+l+1)$.