Advanced Laser
Polarisation effects - Jones' formalism

4.1 Introduction

4.1.1 Maxwell's equations

Starting from the Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.1b}$$

$$\nabla . \mathbf{D} = \rho \tag{4.1c}$$

$$\nabla . \mathbf{B} = 0 \tag{4.1d}$$

associated with the equations for the material:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \tag{4.2a}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.2b}$$

for a dielectric ($\rho = 0$) and non-magnetic medium ($\mathbf{M} = 0$), using the algebraic equation $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla . A) - \nabla^2 \mathbf{A}$ applied to eq. (5.1a), we derive the equation for the propagation of the electric field:

$$\left[\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = \frac{1}{\epsilon_0 c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{4.3}$$

This is a <u>vectorial</u> equation for which the atomic polarisation plays the role of a source term, acting on the electric field.

4.1.2 Atomic polarisation

When an electric field is applied onto a material, it will modify the cloud of electrons around each atom that constitute this material. Such displacement of the electron cloud induces a dipole moment and in the linear case, each contribution (microscopic effect) will add to generate a macroscopic polarisation **P**.

$$\mathbf{E} \;\rightarrow\; \mathbf{p} \;\rightarrow \mathbf{P} = \sum_i \mathbf{p}_i \;\rightarrow \mathbf{E}$$

isotropic medium

From the induced polarisation, we can write the displacement $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi \, \mathbf{E}$. Inserting the polarisation into the equation of propagation (Eq. (5.3)) we have

$$\left[\nabla^2 - \frac{1+\chi}{c^2} \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = 0 \tag{4.4}$$

and the velocity of the wave is $c/(\sqrt{1+\chi}) = c/n$.

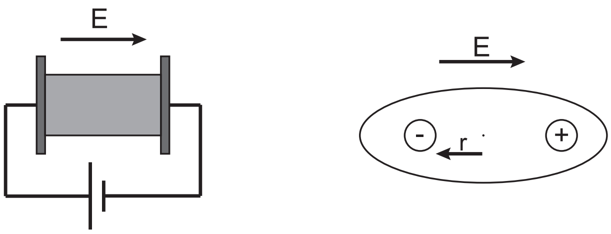


Figure 4.1: Induction of the atomic polarisation by the electric field

dispersion

In the Lorentz model, the electron is bound to the atom, such that the action of the electric field acts like a driving force onto a harmonic oscillator. In this model, the motion of the electron can be described as

$$\ddot{r} + \gamma \dot{r} + \omega_0^2 r = \frac{-qE}{m_e} \tag{4.5}$$

where r is the distance to the centre of charge, γ is a damping coefficient, ω_0 the resonance frequency of the atom, m_e the mass of the electron and q its charge. The solution of the Eq. (5.5) is a superposition of a transient regime (eq. without driving force) with a permanent solution, that follows the driving force. We are only interesting by the permanent regime, and since the incident electric field is $E = E_0 \exp(j\omega t)$, we are seeking solution of the form $r = r_0 \exp(j\omega t)$. Substituting this into Eq. (5.5), we obtain:

$$r_0 = \frac{-q}{m_e\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)} E \tag{4.6}$$

And since the induced microscopic polarisation is given by $p = (-q)r$, we have the macroscopic polarisation

$$P = N(-q)r = \frac{Nq^2}{m_e} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} E \tag{4.7}$$

Since $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(1 + \chi^{(1)})\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E}$, we see from eq. (5.7) than we can get the linear susceptibility $\chi^{(1)}$ or the susceptibility ϵ_r. We can then use the relation

$$\epsilon_r = \left(n + \frac{i\alpha c}{2\omega}\right)^2 \tag{4.8}$$

where n is the refractive index (linked to the real part of the susceptibility $\chi^{(1)}$) and α the absorption, which is linked with the imaginary part of $\chi^{(1)}$. Note that there exists many resonances in the system. Fig. 5.2 shows the evolution of the real part and the imaginary part of χ as a function of ω.

vectorial characteristics of E and B

For a wave described by $E = E_0 e^{j(\omega t - \mathbf{k}.\mathbf{r})}$ and $B = B_0 e^{j(\omega t - \mathbf{k}.\mathbf{r})}$, propagating in a medium without charge nor current ($\rho = 0$, $\mathbf{J} = 0$, the laws of Faraday and Ampere can be rewrite as

$$-i\mathbf{k} \times \mathbf{E} = -i\omega \mathbf{B} \tag{4.9}$$
$$-i\mathbf{k} \times \mathbf{B} = -i\mu_0 \omega \mathbf{D} \tag{4.10}$$

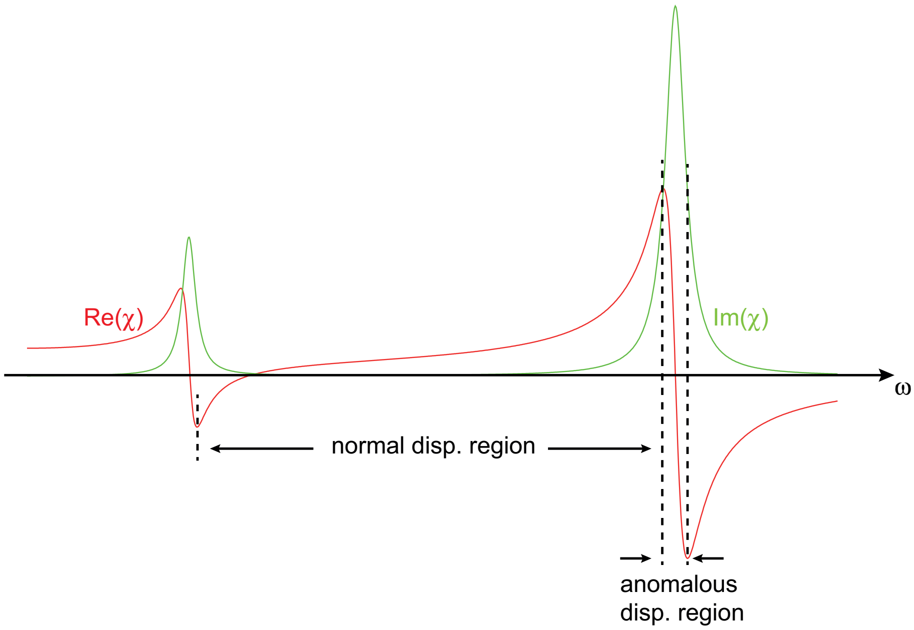


Figure 4.2: Real part and imaginary part of the susceptibility

and in the case of a linear isotropic medium $\mathbf{D} = \epsilon_0(1+\chi)\mathbf{E} \propto \mathbf{E}$, therefore $\mathbf{E} \perp \mathbf{k}$. If z is the direction of propagation (along $\mathbf{k}$) then the electric field is contained in the plane (xOy) and has two components E_x and E_y, which oscillate at a frequency ω, and the resulting electric field is a superposition of both contributions $\mathbf{E} = E_x\mathbf{e_x} + E_y\mathbf{e_y}$ where

$$E_x = A_x \cos(\omega t - kz) \tag{4.11a}$$
$$E_y = A_y \cos(\omega t - kz + \Delta) \tag{4.11b}$$

When E_x and E_y oscillate in phase ($\Delta = 0$), the resulting polarisation is linear, and the orientation with respect to (Ox) (Fig. 5.3). If we call θ the angle to the $x-$axis, then if $A_x = A_y$, $\theta = {}^{\pi}/4$ otherwise θ is defined by $\tan\theta = \left({}^{E_y}/E_x\right)$.

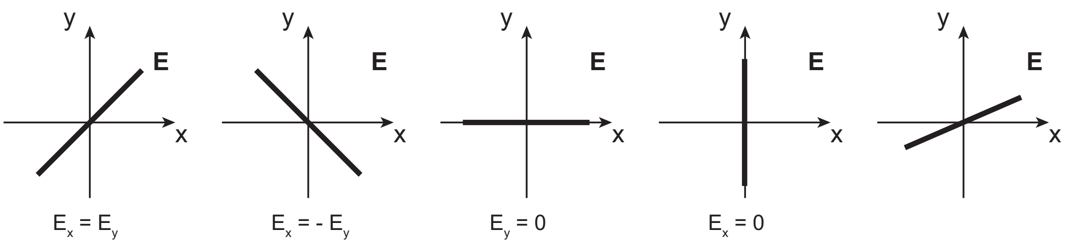


Figure 4.3: Various combination of E_x and E_y leading to linear polarisation.

On the other hand, when E_x and E_y do not oscillate in phase ($\Delta \neq 0$), the projection onto the (xOy) plane becomes an ellipse (Fig. 5.4).

As can be seen from Fig. 5.4, two types of circularly polarised beam can exist depending if E_x is in advance with respect to E_y or delayed. By definition, the beam is *left-hand* circularly polarised if the electric field of the beam coming towards the observer is describing a circle, rotating in the counterclockwise (Fig. 5.5), otherwise it is said to be *right-hand* circularly polarised.

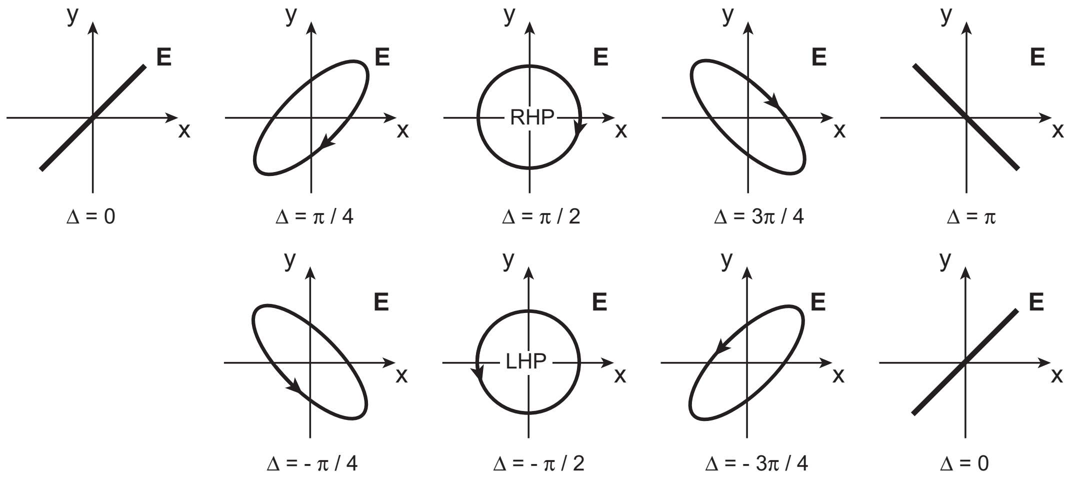


Figure 4.4: Superposition of E_x and E_y, with $|E_x| = |E_y|$ for various dephasing Δ between E_x and E_y.

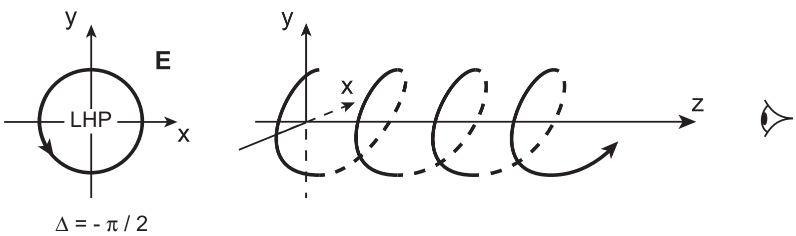


Figure 4.5: Left-hand circularly polarised light.

4.1.3 Refraction and reflection with polarised light

Definition of the incident plane

For what follows it is important to properly define the *incident plane* since this also helps defining the electric field. If the electric field is orthogonal to the incident plane, the wave is defined as TE (*transverse electric* wave) whereas when the electric field belongs to that plane, the wave is defined as TM (*transverse magnetic* wave). As shown on Fig. the wavevector (direction of the propagation of the wave) and the normal of the surface, together with the incident point define the incident plane.

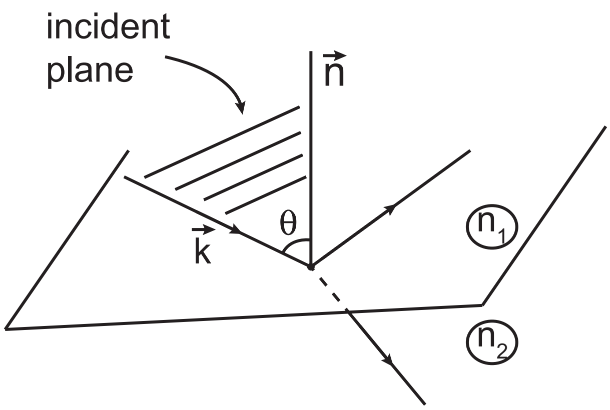


Figure 4.6: Definition of the incident plane.

We can define the reflection and transmission coefficient either in amplitude or in intensity. Defined in intensity, they are directly linked with the conservation of energy and $R+T=1$ where R is the reflection coefficient and T the transmission one. Usually small letter are used to express these coefficients in amplitude and $R=r^2$ and $T^2=(n_1/n_2)\,t^2$.

Depending on the polarisation of the wave the reflection coefficient are given by the *Fresnel's relations*:

$$r_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\,E_\perp \tag{4.12a}$$

$$r_\parallel = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\,E_\parallel \tag{4.12b}$$

Two cases have to be considered depending on the relative magnitude of the refractive indices.

1. $\boxed{n_1 < n_2}$
   At the Brewster angle (i_B), only the TE wave is reflected. Note also that below

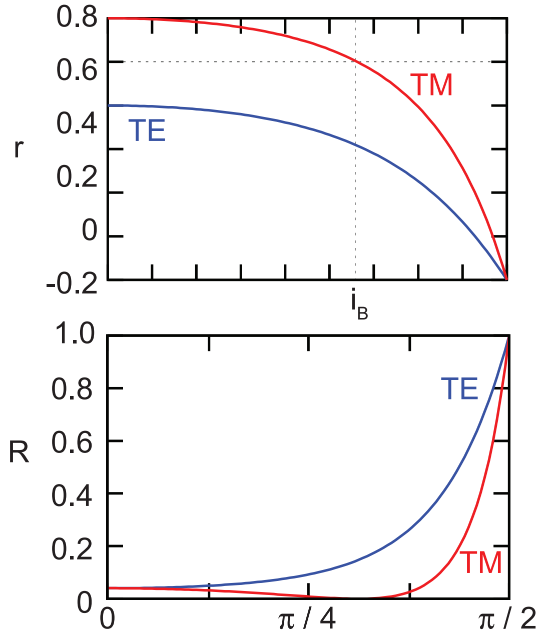


Figure 4.7: Reflections coefficient in amplitude (a) and in intensity (b)

   the TM wave experience a change of phase-shift at the reflection depending if the angle is larger ($\pi-$phase-shift) or smaller (no phase-shift) than the Brewster angle. The TE wave always experience this $\pi-$phase shift at the reflection.

2. $\boxed{n_1 > n_2}$ In this case the situation is very different since there exist a maximum allowed angle: the *critical angle*. Above this angle, there is not refraction possible and the wave is totally reflected. this situation is actually used to coupled light into waveguide by so-called *evanescent coupling*.

4.1.4 Brewster Angle

When a randomly polarised arrives at a surface, if the incident angle is such that $\theta_i+\theta_r=(\pi/2)$ then the polarisation state of the reflected beam can only be transverse electric. If

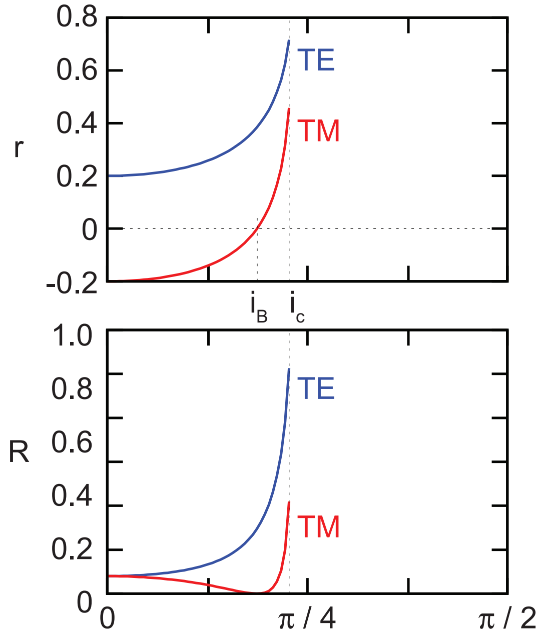


Figure 4.8: Reflections coefficient in amplitude (a) and in intensity (b)

we consider that the electric field create a polarisation that plays the role of the emitter, it is obvious that the emitter for the TM wave cannot emit in the direction of the reflected beam. Only the TE wave is reflected. This is the *Brewster angle*.

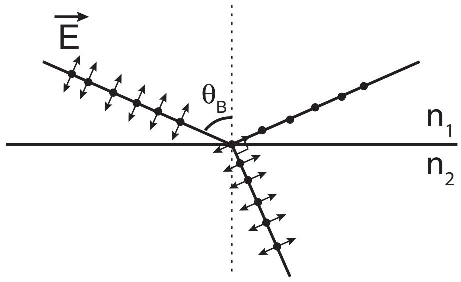


Figure 4.9: Definition of the Brewster angle

4.2 Non isotropic medium

So far, we have only consider that the medium where the electric field propagated was isotropic, and therefore the dielectric susceptibility χ was the same for any direction. In reality, this may be no true, and the displacement $\mathbf{D} = \epsilon_0(1+\chi)\mathbf{E} = \epsilon_0\epsilon(r)\mathbf{E}$ now involves a tensor for $\epsilon(r)$:

$$\mathbf{D} = \epsilon_0 \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{33} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \mathbf{E} \tag{4.13}$$

Mathematically, this matrix can be diagonalized, and this corresponds physically to find the principal axis (eigenvectors) of the medium. We can then write:

$$\mathbf{D} = \epsilon_0 \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \mathbf{E} = \epsilon_0 \hat{\epsilon_r} \mathbf{E} \tag{4.14}$$

where the tensor $\hat{\epsilon_r}$ consists in $\epsilon_i = n_i^2$, with $i = (x, y, z)$. From this tensor, it clear that there exists three situations, three types of material. For an isotropic material, $\forall i, n_i = \text{cste}$. On the other hand, media in which all the three dielectric constants ϵ_i are difference are called *bi-axial*, whereas those in which only two of the three are different are called *uni-axial*.

In the general case, $\mathbf{k} \cdot \mathbf{E} \neq 0$ then the algebraic relation $\nabla \times \nabla \times \nabla E = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ applied to the Faraday law equation (Eq. (5.1a) leads to

$$\mathbf{k}^2 \mathbf{E} - \mu_0 \omega^2 \hat{\epsilon}\ \mathbf{E} = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) \tag{4.15}$$

and considering the vectorial nature of both $\mathbf{k}$ and $\mathbf{E}$ we have Eq. (45):

$$\begin{aligned} \left(k_x^2 + k_y^2 + k_z^2\right) (E_x \mathbf{e_x} + E_y \mathbf{e_y}\ +\ E_z \mathbf{e_z}) - \mu_0 \omega^2 \epsilon_0 \left(n_x^2 E_x \mathbf{e_x} + n_y^2 E_y \mathbf{e_y} + n_z^2 E_z \mathbf{e_z}\right) \\ = (k_x \mathbf{e_x} + k_y \mathbf{e_y} + k_z \mathbf{e_z}) (k_x \mathbf{e_x} + k_y \mathbf{e_y} + k_z \mathbf{e_z}) \\ = k_x^2 E_x \mathbf{e_x} + k_x k_y E_y \mathbf{e_y} + k_x k_z E_z \mathbf{e_z} \\ + k_x k_y E_x \mathbf{e_x} + k_y^2 E_y \mathbf{e_x} + k_x k_z E_z \mathbf{e_z} \\ + k_x k_z E_X \mathbf{e_x} + k_y k_z E_z \mathbf{e_y} + k_z^2 E_z \mathbf{e_z} \end{aligned} \tag{4.16}$$

and on each axis:

$$/\mathbf{e_x}\ :\quad \left(k_y^2 + k_z^2 - \frac{n_x^2 \omega^2}{c^2}\right) E_x = k_x k_y E_y + k_x k_z E_z \tag{4.17a}$$

$$/\mathbf{e_y}\ :\quad \left(k_x^2 + k_z^2 - \frac{n_y^2 \omega^2}{c^2}\right) E_y = k_y k_x E_x + k_y k_z E_z \tag{4.17b}$$

$$/\mathbf{e_z}\ :\quad \left(k_x^2 + k_y^2 - \frac{n_z^2 \omega^2}{c^2}\right) E_z = k_z k_x E_x + k_z k_y E_y \tag{4.17c}$$

These equations have a solution if and only if[1]

$$Det. = \begin{vmatrix} \left(k_y^2 + k_z^2 - \frac{n_x^2 \omega^2}{c^2}\right) & -k_x ky & -k_x k_z \\ -k_x k_y & \left(k_x^2 + k_z^2 - \frac{n_y^2 \omega^2}{c^2}\right) & -k_y k_z \\ -k_x k_z & -k_y k_z & \left(k_x^2 + k_y^2 - \frac{n_z^2 \omega^2}{c^2}\right) \end{vmatrix} = 0 \tag{4.18}$$

After calculation, we obtain:

$$\begin{aligned} Det. = \frac{-\omega^4}{c^4} + \frac{\omega^2}{c^2} \left(\frac{k_y^2 + k_z^2}{n_x^2} + \frac{k_x^2 + k_z^2}{n_y^2} + \frac{k_x^2 + k_y^2}{n_z^2}\right) \\ + \left(\frac{k_x^2}{n_y^2 n_z^2} + \frac{k_y^2}{n_x^2 n_z^2} + \frac{k_z^2}{n_x^2 n_y^2}\right) \left(k_x^2 + k_y^2 + k_z^2\right) = 0 \end{aligned} \tag{4.19}$$

[1] we remind that such determinant is calculated by:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

uni-axial crystal

In the general case, the three components (n_x, n_y, n_z) are different, and we then refer these crystal to *bi-axial* crystal. But there is a simpler case, we , $n_x = n_y = n_o = n_\perp$ (the ordinary axis), and $n_z = n_e = n_\parallel$ (the extraordinary index). In this case, the Eq. (5.20) becomes

$$Det. = \frac{-\omega^4}{c^4} + \frac{\omega^2}{c^2}\left(\frac{k_y^2+k_z^2}{n_o^2} + \frac{k_x^2+k_z^2}{n_o^2} + \frac{k_x^2+k_y^2}{n_e^2}\right) + \left(\frac{k_x^2}{n_o^2 n_e^2} + \frac{k_y^2}{n_o^2 n_e^2} + \frac{k_z^2}{n_o^4}\right)\left(k_x^2+k_y^2+k_z^2\right) = 0 \quad (4.20)$$

which can be factorised as

$$\left(\frac{k_x^2+k_y^2+k_z^2}{n_o^2} - \frac{\omega^2}{c^2}\right)\left(\frac{k_x^2+k_y^2}{n_e^2} + \frac{k_z^2}{n_o^2} - \frac{\omega^2}{c^2}\right) = 0 \quad (4.21)$$

The first term of this equation corresponds to a sphere whereas the second represents an ellipsoid. The difference between the refractive indices of the ordinary and the extraordinary beam is known as the *birefringence* $\Delta n = n_e - n_o$. Depending of its sign, the crystal is said to be *positive* or *negative* (Fig. 5.10).

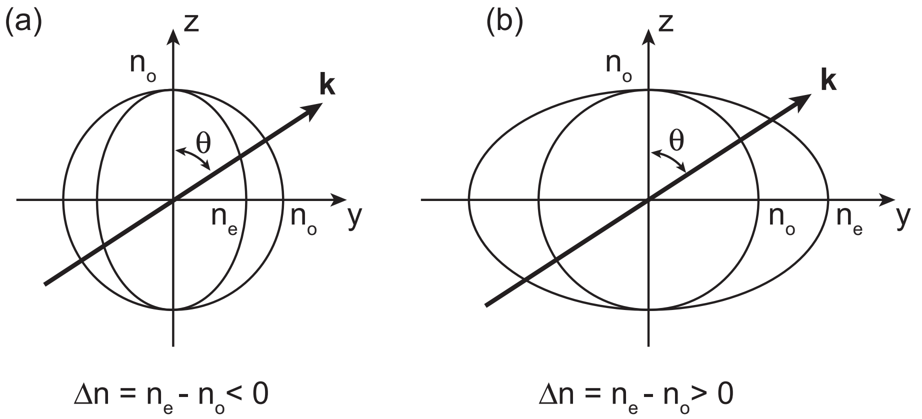


Figure 4.10: Representation of the *index ellipsoid* cut in the plane (O_x) for (a) negative and (b) positive crystal. **k** coincides with the optical axis.

4.3 Jones' formalism - representation of the polarisation

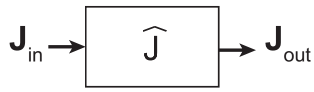
Since the electric field is transverse, it can simply written as

$$\mathbf{E}(z,t) = \begin{pmatrix} A_x\cos(\omega t - kz + \delta_x) \\ A_y\cos(\omega t - kz + \delta_y) \\ 0 \end{pmatrix} = \text{Re}\left[\begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i(\omega t - kz)}}\right] \quad (4.22)$$

In order to see the evolution of the polarisation state as the wave is propagating, Jones established in 1941 a useful formalism, in which the polarisation field is simply describe as

$$\mathbf{J} = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{1}{\sqrt{A_x^2 + A_y^2}} \begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{pmatrix} \tag{4.23}$$

The norm of the Jones' matrix is $JJ^* = 1$. Using this formalism, the polarisation of the incident beam can be described and follow as the beams goes through the various element that may affect the polarisation. Each element will be represented by a 2×2 matrix:



$$\text{with} \quad \begin{cases} J^x_{\text{out}} &= A\ J^x_{\text{in}} + B\ J^y_{\text{in}} \\ J^y_{\text{out}} &= C\ J^x_{\text{in}} + D\ J^y_{\text{in}} \end{cases}$$

Particular states of polarisation

The different states of polarisation can be expressed as

linear polarisation

$$\mathbf{E} = E_x \mathbf{e_x} \Longrightarrow \mathbf{J} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{E} = E_y \mathbf{e_y} \Longrightarrow \mathbf{J} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\textit{linear at } 45^o \quad \mathbf{E} = E_0\ \mathbf{e_x} + E_0\ \mathbf{e_y} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\textit{linear at } -45^o \quad \mathbf{E} = E_0\ \mathbf{e_x} - E_0\ \mathbf{e_y} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

circular polarisation

$$|E_x| = |E_y| \text{ and } (\delta_{\text{x}} - \delta_{\text{y}}) = \pm\frac{\pi}{2} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

4.3.1 basics polarising element & Jones calculus

Polariser

A polariser is an element which produces linear polarisation from any arbitrary polarisation states. Somehow, this is a simple projection of the polarisation state on the axis of the polariser. If the polariser is along $\mathbf{e}_x$ (resp. $\mathbf{e}_y$) then the Jones' matrix is

$$\hat{\mathbf{J}}_{\mathbf{P}\|\mathbf{e_x}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{\mathbf{J}}_{\mathbf{P}\|\mathbf{e_y}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.25}$$

If the polariser is at $\pm 45^o$ then:

$$\hat{\mathbf{J}}_{\mathbf{P}[+45^o]} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \hat{\mathbf{J}}_{\mathbf{P}[-45^o]} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4.26}$$

Use of a polariser

1. Let's assume that we have a beam that is linearly polarised along the $\mathbf{e}_x$ axis and the axis of the polariser is set at 45^o:

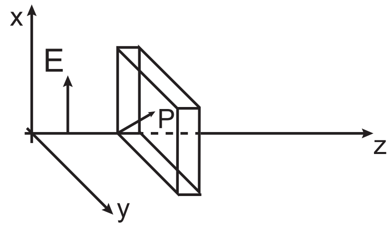


Figure 4.11: action of a polariser set at 45^o

The resulting polarisation state can be readily calculated by

$$J_{\text{out}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.27}$$

The resulting intensity is $\left(1/2\right) E_0^2$: half of the intensity is lost in the process.

2. Another interesting case is when we use two polarisers orthogonal to each other, in a so-called *crossed* configuration. The resulting transfer matrix is

$$\hat{J}_{\text{crossed}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.28}$$

No light can pass through!

3. In the general case where a polariser is set at an angle α then the resulting polarisation state can be derived using the rotational transformation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

then the final polarisation state can be calculated as

$$J_{\text{out}} = R(-\alpha)\; \hat{\mathbf{J}}_{P\|\mathbf{e}'_\mathbf{x}}\; R(\alpha) J_{\text{in}} \tag{4.29}$$

Wave-plate - retarder

Let us now assume that we insert on the path of the beam a plate made of a birefringent material. The plate is oriented such that it optic axis is along $\mathbf{e_x}$ (Fig. 5.12). Within such a configuration the refractive index along the $x-$axis is $\mathrm{n}_{\|}$ and the one along the $y-$axis is $\mathrm{n}_{\perp}$.
The wave incident to that plate is defined as $E = E_0 \exp i\left(\omega_0 t - kz + \varphi_0\right)$. After the propagation in the birefringent material the transverse component of the electric fields have experienced a different phase shift so that

$$E_x(z=d) = E_{0x} \exp i\left(\omega t - k_0 n_{\|} d + \varphi_{0x}\right) \tag{4.30a}$$

$$E_y(z=d) = E_{0y} \exp i\left(\omega t - k_0 n_{\perp} d + \varphi_{0y}\right) \tag{4.30b}$$

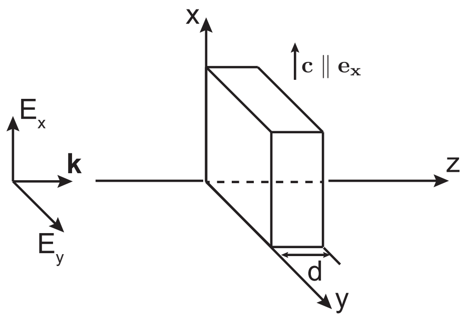


Figure 4.12: Schematic of the setup using a wave-plate. The optic axis **c** is co-linear with $\mathbf{e_x}$. The thickness of the plate is d.

At the output of the wave-plate the phase different between the $x-$component and the $y-$component of the electric field is

$$\Delta\varphi = \Delta\varphi_0 + (k_0 n_{\parallel} - k_0 n_{\perp})d \tag{4.31}$$

where $\Delta\varphi_0 = \varphi_{0y} - \varphi_{0x}$ is the phase different between the components E_x and E_y at the input of the wave-plate. Note that since in a birefringent material $n_{\parallel} \neq n_{\perp}$ then the phase shift induced during the propagation is not the same on both axis. For this reason such optical element is sometimes called a *retarder*.

Jones matrix for a wave-plate

Considering that the induced phase for each transverse component of the electric field is now determined (eq. (5.30a)) we can readily translate this into Jones' formalism:

$$\mathbf{J}_{\text{out}} = \begin{pmatrix} E_x e^{-ik_0 n_{\parallel} d} \\ E_y e^{-ik_0 n_{\perp} d} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-ik_0 n_{\parallel} d} & 0 \\ 0 & e^{-ik_0 n_{\perp} d} \end{pmatrix}}_{\hat{J}} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{4.32}$$

We can also write the Jones' matrix in a more symmetric way as

$$\hat{J} = e^{i\frac{\Delta\varphi'}{2}} \begin{pmatrix} e^{-i\frac{\Delta\varphi'}{2}} & 0 \\ 0 & e^{i\frac{\Delta\varphi'}{2}} \end{pmatrix} \tag{4.33}$$

where $\Delta\varphi'$ is phase shift induced

$$\Delta\varphi' = k_0(n_{\parallel} - n_{\perp})d = k_0 d\delta n \tag{4.34}$$

half-wave plate

Let us now assume that the retardation is π. For an electric field polarised at 45^o (*i.e.* $|E_x| = |E_y|$ and $\Delta\varphi_0 = 0$) then

$$\Delta\varphi' = d\frac{2\pi}{\lambda}\delta n = \pi + 2m\pi \qquad \text{with } m \in \mathbb{Z}^* \tag{4.35}$$

$$\Rightarrow\ d \cdot \delta n = \frac{\lambda}{2}(1 + 2m) \tag{4.36}$$

Such a plate is called a *half-wave plate*. m is called the order of the plate. In practise δn is determined by the material but the thickness of the plate can be chosen. For $m = 0$ we talk about *zero-order* wave-plate. They are usually more expansive and very fragile. According to the eq. (5.33) the Jones' matrix for a half-wave plate is

$$\hat{J}_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.37}$$

Use of a half-wave plate. Note that we calculate the influence of the wave-plate in a very particular case: the input beam is polarised along the optic axis of the wave-plate. In general this is not the case. Let us assume that the axis of the plate is rotated by an angle α with respect to the linear polarisation of the input beam (Fig.). We should not use the eq. (5.29). In other word we need first to align the electric field with the optic axis of the wave-plate, then apply the wave-plate and return in the original frame of work:

$$J'_{in} = R(\alpha)J_{in} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ 0 \end{pmatrix} = \begin{pmatrix} E\cos\alpha \\ -E\sin\alpha \end{pmatrix} \tag{4.38}$$

and after the wave-plate

$$J'_{out} = \begin{pmatrix} E\cos\alpha \\ E\sin\alpha \end{pmatrix} \tag{4.39}$$

and finally

$$J_{out} = R(-\alpha)J'_{out} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E\cos\alpha \\ E\sin\alpha \end{pmatrix} = E \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} \tag{4.40}$$

The beam is still linearly polarised but the orientation of its polarisation has been rotated by 2α!

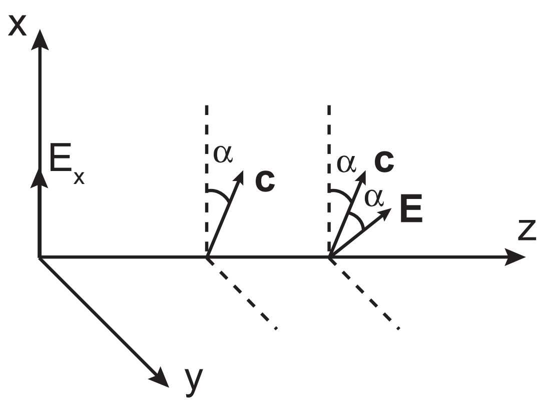


Figure 4.13: Effect of a half-wave plate, for which the optic axis **c** makes an angle α with the direction of the electric field incident on the wave-plate.

quarter-wave plate

Another important optical element is the quarter-wave plate. By contrast with the half-wave plate the induced phase shift is now ($\pi/2$). A wave linearly polarised at 45^o (*i.e.*

$|E_x| = |E_y|$ and $\Delta\varphi_0 = 0$) will transform into a circularly polarised wave according Fig. 5.4. Using the eq. (5.34) we can readily obtain

$$d \cdot \delta n = \frac{\lambda}{4}(1+2m) \qquad \text{with } m \in \mathbb{Z}^* \tag{4.41}$$

and the Jones' matrix is

$$\hat{J}_{\lambda/4} = e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{4.42}$$

As for the half-wave plate the orientation of the quarter-wave plate is important. If the linearly polarised incident beam is such that its electric field is along (or orthogonal to) the optic axis of the quarter-wave plate then the plate has no influence. By contrast it the input field is linearly polarised at 45^o then the situation is very different:

$$\mathbf{J}_{out} = e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{4.43}$$

The beam is indeed circularly polarised.

4.3.2 Representation of polarisation state: the Poincare's sphere

Fig. 5.4 represents all the possible polarisation state (besides for unpolarized light). One way to determine precisely the state of polarisation of a beam is to use the four *Stokes parameters*, introduced by G.G. Stokes in 1852. The main advantage of these parameters is that they can be fully determine through intensity measurement, and can therefore be straightforwardly been measured. With the definition of the field (Eq. (5.11)), the Stokes parameters are

$$s_0 = |\mathbf{E}.\mathbf{e_x}|^2 + |\mathbf{E}.\mathbf{e_y}|^2 = A_x^2 + A_y^2 \tag{4.44a}$$

$$s_1 = |\mathbf{E}.\mathbf{e_x}|^2 - |\mathbf{E}.\mathbf{e_x}|^2 = A_x^2 - A_y^2 \tag{4.44b}$$

$$s_2 = 2\mathrm{Re}\left[(\mathbf{E}.\mathbf{e_x})^*(\mathbf{E}.\mathbf{e_y})\right] = 2\mathrm{A_x A_y}\cos\Delta \tag{4.44c}$$

$$s_3 = 2\mathrm{Im}\left[(\mathbf{E}.\mathbf{e_x})^*(\mathbf{E}.\mathbf{e_y})\right] = 2\mathrm{A_x A_y}\sin\Delta \tag{4.44d}$$

Note that $s_0^2 = s_1^2 + s_2^3 + s_3^2$ is the total intensity of the wave, s_1 gives the preponderance of $x-$linear polarisation over the $y-$polarisation, and s_2 and s_3 give the phase information. Finally, one way to visualise the Stokes parameter is to use the *Poincare sphere* introduce by Henri Poincare in 1892 (Fig. 5.14).

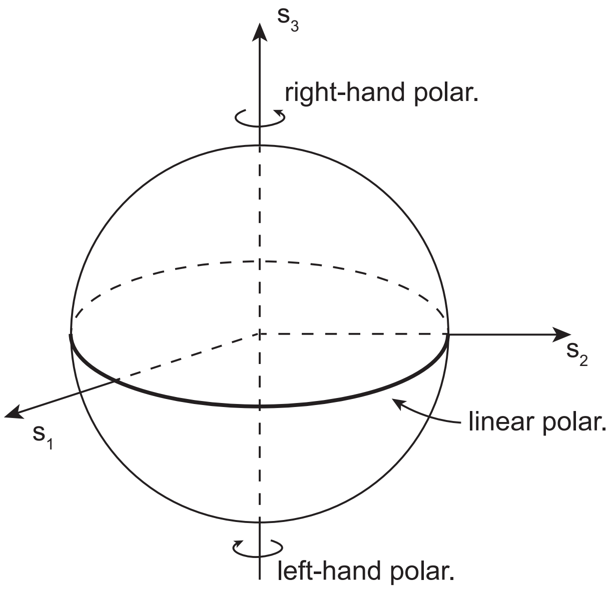


Figure 4.14: Poincare's sphere representing the possible states of polarisation.

.1 index ellipsoid - *alternative calculation*

Since in the general case, $\mathbf{k}.\mathbf{E} \neq 0$ then the algebraic relation $\nabla\times\nabla\times\nabla E = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E}$ applied to the Faraday law equation (Eq. (5.1a) leads to

$$\mathbf{k}^2\mathbf{E} - \mu_0\omega^2\hat{\epsilon}\ \mathbf{E} = \mathbf{k}(\mathbf{k}.\mathbf{E}) \tag{45}$$

Therefore on the axis (O_x):

$$\begin{aligned} /(O_x) \qquad & \mathbf{k}^2 E_x - \mu_0\omega^2 D_x = k_x(\mathbf{k}.\mathbf{E}) \\ \Leftrightarrow \quad & \mathbf{k}^2\frac{D_x}{\epsilon_1} - \mu_0\omega^2 D_x = k_x(\mathbf{k}.\mathbf{E}) \\ \Leftrightarrow \quad & \frac{D_x}{\epsilon_1}\left(\mathbf{k}^2 - k_1^2\right) = k_x(\mathbf{k}.\mathbf{E}) \end{aligned} \tag{46}$$

Since the medium is a dielectric (no charge), the Maxwell-Gauss equation $\nabla.\mathbf{D} = 0$ leads to $\sum_{i=x,y,z} k_i D_i = 0$:

$$\epsilon_1\frac{k_x^2}{k^2 - k_1^2} + \epsilon_2\frac{k_y^2}{k^2 - k_2^2} + \epsilon_3\frac{k_z^2}{k^2 - k_3^2} = 0 \tag{47}$$

which can then be rewritten as

$$n_1^2\frac{n_x^2}{n^2 - n_1^2} + n_2^2\frac{n_y^2}{n^2 - n_2^2} + n_3^2\frac{n_z^2}{n^2 - n_3^2} = 0 \tag{48}$$

This is an ellipsoid and it characterises the anisotropic medium. Such surface is actually hard to visualise, especially in the general case, where the three semi-axis of the *index ellipsoid* n_1, n_2, n_3 are different. One way to represent this ellipsoid is actually to cut it by the (O_{yz}) plane ($n_x = 0$). Note that to safely do $n_x = 0$ we need to consider both cases: (i) $n^2 = n_1^2$ and (ii) $n^2 \neq n_1^2$.

$\boxed{n^2 - n_1^2 \neq 0}$

Eq. (48) can simply be written as:

$$
\begin{aligned}
& n_2^2 n_y^2 \left(n^2 - n_3^2\right) + n_3^2 n_z^2 \left(n^2 - n_2^2\right) = 0 \\
& \quad\Leftrightarrow \quad n_2^2 n_y^2 n^2 + n_3^2 n_z^2 n^2 = n_2^2 n_3^2 n_y^2 + n_3^2 n_2^3 n_z^2 = \left(n_y^2 + n_z^2\right) n_2^2 n_3^2 \\
& \quad\Leftrightarrow \quad \frac{n^2 n_y^2}{n_3^2} + \frac{n_z^2 n^2}{n_2^2} = n_y^2 + n_z^2
\end{aligned}
\tag{49}
$$

and since $n^2 = n_x^2 + n_y^2 + n_z^2$ we obtain

$$
\frac{n_y^2}{n_3^2} + \frac{n_z^2}{n_2^2} = 1 \tag{50}
$$

This is the equation of an ellipse!

$\boxed{n^2 - n_1^2 = 0}$

Similar calculation leads this time to the equation of a circle

$$
n_y^2 + n_z^2 = n_1^2 \tag{51}
$$