
Advanced Laser

EFFECTS OF DISPERSION ON PULSES

0.1 Introduction

As a short pulse propagates through a material it may experience the effect of dispersion. As a result it will (in general) broaden. The idea of this chapter is to describe the effect of dispersion on the propagating pulse and to propose strategies to balance such effect in order to maintain the *duration* and *shape* of the pulse. The overall dispersion of the cavity will determine the ultimate duration of the pulse exiting the laser cavity.

0.2 Propagation in a dispersion medium

0.2.1 Plane wave

First we start with the simple case of a plane wave propagating along the z-direction. To remain as simple as possible the polarisation of the wave is linear. It is mathematically described by

$$E(z, t) = A_0 \cos(\omega t - kz) = \text{Re} [A_0 e^{-i(\omega_0 t - kz)}] = \frac{1}{2} A_0 e^{-i(\omega_0 t - kz)} + c.c. \quad (1)$$

where A_0 is the amplitude of the wave, ω_0 its angular frequency and k its wave-number. The total phase acquired by the wave is

$$\phi = \omega_0 t - kz \quad (2)$$

We can readily define the speed of the phase-front of wave by setting $d\phi = 0$ which is $\omega_0 dt = kdz$. The *phase-velocity* is then

$$v_\phi = \frac{dz}{dt} = \frac{\omega_0}{k} \quad (3)$$

Note that the phase velocity is also (c/n) we can write the equation for the phase-velocity as

$$k_0(\omega) = \frac{\omega_0 n}{c} \quad (4)$$

which is commonly known as the *dispersion relation*.

0.2.2 Pulse

In the context of mode-locking theory we already saw that a pulse can be described as the coherent superposition of the longitudinal modes oscillating in the cavity. In the previous semester we used the idea that all the modes contribute equally to the final pulse (Fig. 1.a) but in reality the profile of the gain (Fig. 1.b) has to be taken into account.

Using the gain profile a few characteristics of the pulses:

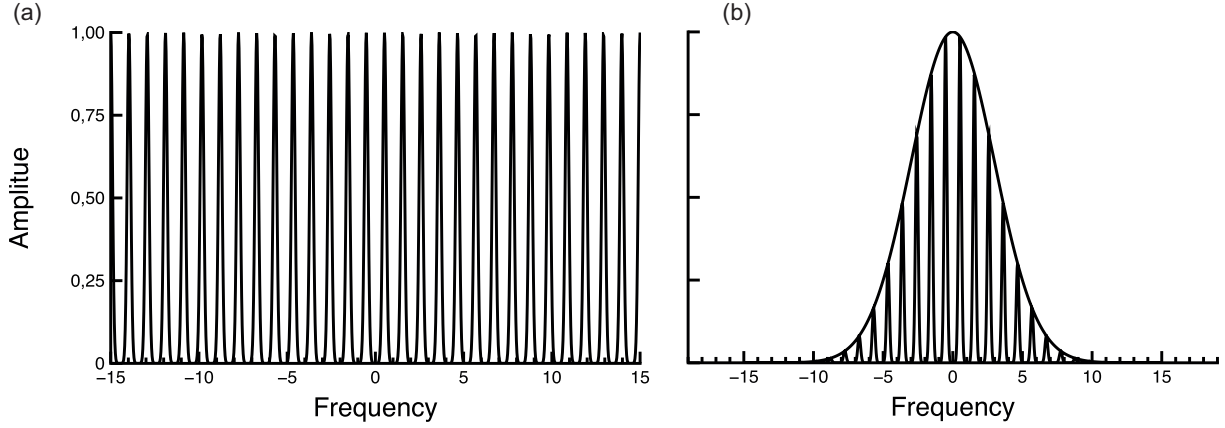


Figure 1: Longitudinal modes without (a) any gain information and (b) with gain profile.

- its carrier frequency is ω_0
- its bandwidth is $\Delta\omega$

Of course this pulse propagates in a material, which is described by its refractive index $n = n(\omega)$. At any given time and position the field is expressed as a Fourier expansion in plane waves

$$E(z, t) = \int_{-\infty}^{+\infty} \tilde{E}(\omega) e^{-i(\omega t - k z)} \frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} \tilde{A}(\omega - \omega_0) e^{-i(\omega t - k z)} d\omega \quad (5)$$

where $A(\omega - \omega_0)$ is the complex amplitude of each spectral component. In this integral the wave-number k is also a function of the angular frequency $k = k(\omega - \omega_0)$. Let's assume that, over the bandwidth of the pulse, the dispersion relation $k(\omega)$ can be approximated by a Taylor expansion truncated at the first order:

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega} \right)_{\omega_0} (\omega - \omega_0) + o(\omega^2) \quad (6)$$

where $k_0 = k(\omega_0)$ and the subscript in the derivative indicates that the derivative is calculated at $\omega = \omega_0$. Inserting this relation of dispersion into eq. (5) yields

$$\begin{aligned} E(z, t) &= \int_{-\infty}^{\infty} \tilde{A}(\omega - \omega_0) e^{-i\left\{ \omega t - \left[k_0 + \left(\frac{dk}{d\omega} \right)_{\omega_0} (\omega - \omega_0) \right] z \right\}} d\omega \\ &= \int_{-\infty}^{\infty} \tilde{A}(\omega - \omega_0) e^{-i\left\{ (\omega - \omega_0 + \omega_0) t - \left[k_0 + \left(\frac{dk}{d\omega} \right)_{\omega_0} (\omega - \omega_0) \right] z \right\}} d\omega \\ &= e^{-i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \tilde{A}(\omega - \omega_0) e^{-i\left\{ (\omega - \omega_0) t - \left[\left(\frac{dk}{d\omega} \right)_{\omega_0} (\omega - \omega_0) \right] z \right\}} d\omega \\ E(z, t) &= e^{-i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \tilde{A}(\Omega) e^{-i\left\{ \Omega t - \left[\left(\frac{dk}{d\omega} \right)_{\omega_0} \Omega \right] z \right\}} d\Omega \end{aligned} \quad (7)$$

with $\Omega = \omega - \omega_0$. Note that this integral is simply the Fourier transform¹ of the spectral

¹We remind that the Fourier transform is defined as

$$G(\omega) = \mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt \quad \text{and} \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} d\omega$$

amplitude $A(\Omega)$:

$$E(z, t) = A \left[t - \left(\frac{z}{v_G} \right) \right] e^{-i(\omega_0 t - k_0 z)} \quad (8)$$

where we have defined the group velocity² v_G by

$$\frac{1}{v_G} = \left(\frac{dk}{d\omega} \right)_{\omega_0} \quad (9)$$

The group velocity can be interpreted as the speed at which the wave-packet propagates. From the relation of dispersion this is also obvious that this is the inverse of the slope of this relation. As we see for the eq. (8), the pulse does not change its shape but acquires a delay as it propagates along z . After a distance L in the dispersive medium the pulse has accumulated a delay

$$\tau_G = \frac{L}{v_G} \quad (10)$$

and a phase-shift $\phi(\Omega) = L k(\Omega)$. From the definition of the group velocity (eq. (9)) we see that the *group delay* is defined as

$$\tau_G = \phi'(\Omega) \quad (11)$$

0.2.3 Group velocity dispersion

In the previous section, we truncated the Taylor expansion at the first order. This corresponds to the dashed line on Fig. 2. Although this is perfectly valid for the red spectrum on that figure, it is no longer the case for the larger spectrum (green on the figure), corresponding to a pulse with a shorter duration³.

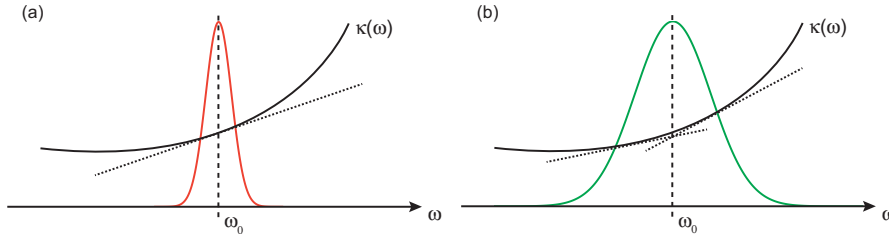


Figure 2: Relationship between the spectrum of the pulse and the normalized dispersion relation $\kappa(\omega)$ in the case of a long (a) and a short (b) pulse.

Since the group velocity is defined as the inverse of the slope of the relation of dispersion (dashed line on Fig. 2), it is clear that in the case of the short-duration pulse

We also remind the shift properties of the Fourier transform:

$$\mathcal{F}[g(t - \tau)] = e^{i\omega\tau} \mathcal{F}[g(t)] = e^{i\omega\tau} G(\omega)$$

²The group velocity is in general different from the phase velocity.

³The curve corresponding to $k(\omega) = n(\omega)\omega/c$ does not vary very much. We plotted here the normalized dispersion relation $\kappa(\omega) = k(\omega) - \left(\frac{\omega k(\omega_{\max})}{\omega_{\max}} \right)$. Regarding the slope (dashed line on the Fig. 2), this only adds a constant value to the derivative of $k(\omega)$ with respect to ω and therefore does not affect the quantitative description of the dispersion since this constant will vanish in eq. (12).

(broad spectrum) the group velocity that is calculated at the edge of the spectrum are not identical since the slopes at different from each others. In others words, as the pulse propagates different part of its spectrum travel at different group velocity, yield a *dispersion of the group-delay* $\Delta\tau_G$. This can be evaluated from the fastest (with frequency ω_1) and the slowest (with a frequency ω_2) spectral component:

$$\Delta\tau_G = L \left(\frac{1}{v_G^{(2)}} - \frac{1}{v_G^{(1)}} \right) = L \left[\left(\frac{dk}{d\omega} \right)_{\omega_2} - \left(\frac{dk}{d\omega} \right)_{\omega_1} \right] \quad (12)$$

If we consider that the group-velocity $v_G^{(1)}$ is not too different from $v_G^{(2)}$ then we can use a Taylor expansion to re-calculate $\Delta\tau_G$

$$\Delta\tau_G = L \left[\frac{1}{v_G(\omega_2)} - \frac{1}{v_G(\omega_1)} \right] = L \left[\frac{1}{v_G(\omega_1)} + \frac{d}{d\omega} \left(\frac{1}{v_g} \right)_{\omega_1} (\omega_2 - \omega_1) - \frac{1}{v_G(\omega_1)} \right] \quad (13)$$

And since

$$\frac{d}{d\omega} \left(\frac{1}{v_G} \right) = \frac{d}{d\omega} \left(\frac{dk}{d\omega} \right) = \frac{d^2k}{d\omega^2} \quad (14)$$

we can rewrite broadening $\Delta\tau_G$ as

$$\Delta\tau_G = L \underbrace{\left[\left(\frac{d^2k}{d\omega^2} \right)_{\omega_0} \right]}_{\text{GVD}} (\omega - \omega_0) \quad (15)$$

As previously we can use the derivative of the phase and write

$$\Delta\tau_G = \phi''(\omega_0)(\omega - \omega_0) \quad (16)$$

where $\phi''(\omega_0)$ is often called the *group-delay dispersion* (GDD). Considering eq. (11) we see that we are actually doing a Taylor expansion of the equation for the group delay. Including higher-order term this is

$$\tau_G = \phi'(\omega_0) + \phi''(\omega - \omega_0) + \frac{1}{2} \underbrace{\phi'''(\omega - \omega_0)^2}_{\text{TOD}} + o(\omega^3) \quad (17)$$

where TOD is the influence of *third order dispersion*. Regarding the units of the different variable that we just introduced. We see from eq. (15) that the *group velocity dispersion* (GVD) has the unit of $[\text{time}]^2/[\text{length}]$ whilst the group delay dispersion has the unit of $[\text{time}]^2$. In practise the GDD is mostly used to characterise the dispersion of mirror, where the parameter L does not have much sense. It is usually measured in fs^2 .

It is interesting to point that the impossibility to use the linear expansion of the dispersion relation yields the introduction of the second derivative $\left(\frac{d^2k}{d\omega^2} \right)$. It was actually clear from the beginning that our initial Taylor expansion was no longer valid and we had to introduce the next term in the definition of $k(\omega)$:

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega} \right)_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left(\frac{d^2k}{d\omega^2} \right)_{\omega_0} (\omega - \omega_0)^2 + o(\omega^3) \quad (18)$$

It is now obvious to associate the second order term in this expansion to the GVD.

0.3 Influence of dispersion on a pulse

0.3.1 Equation of propagation

To see the evolution of this pulse ($E(z, t)$), for which we only have the envelope we need to derive a propagation equation. Actually since the shape of the pulse is supposed to change slowly, we can assume that a small change in frequency $\Delta\omega = \omega - \omega_0$ will simply correspond to a change a wave-number $\Delta k = k - k_0$ so that the Fourier transform of the envelope $E(z, t)$ can be written as

$$\tilde{E}(\Delta k, \Delta\omega) = \iint_{-\infty}^{\infty} E(x, t) e^{i(\Delta\omega t - \Delta k z)} dz dt \quad (19)$$

and the inverse Fourier transform as

$$E(z, t) = \iint_{-\infty}^{\infty} \tilde{E}(\Delta k, \Delta\omega) e^{-i(\Delta\omega t - \Delta k z)} d(\Delta\omega) d(\Delta k) \quad (20)$$

We can then identify that

$$\frac{\partial E}{\partial t} = -i\Delta\omega E(z, t) \quad (21a)$$

$$\frac{\partial E}{\partial z} = +i\Delta k E(z, t) \quad (21b)$$

$$(21c)$$

In other words, we can replace in the relation of dispersion $\Delta\omega$ and Δk can be replaced by the operators:

$$\Delta\omega = (\omega - \omega_0) \longleftrightarrow +i\frac{\partial}{\partial t} \quad (22a)$$

$$\Delta k = (k - k_0) \longleftrightarrow -i\frac{\partial}{\partial z} \quad (22b)$$

$$(22c)$$

Therefore the dispersion relation (eq. (18)) yields

$$-i\frac{\partial}{\partial z} = ik_1\frac{\partial}{\partial t} - \frac{1}{2}k_2\frac{\partial^2}{\partial t^2} \quad (23)$$

where we replaced the i^{th} derivative of $k(\omega)$ with respect to ω by $k_i(\omega)$. We can apply this equation to our envelope pulse $E(z, t)$ to get

$$i\left[\frac{\partial}{\partial z} + k_1\frac{\partial}{\partial t}\right] E = \frac{1}{2}k_2\frac{\partial^2 E}{\partial t^2} \quad (24)$$

As we saw previously (eq. (8)), if the bandwidth of the pulse is small enough that the second order dispersion k_2 can be neglected the equation (24) becomes trivial and its solution can expressed as any function of the variable $z - t/k_1$. This corresponds to the situation when the pulse⁴ propagates at the group velocity⁵ without changing its shape.

⁴We are here considering pulses, but the eq. (24) is very general and we could imagine other types of functions.

⁵We remind that $k_1 = \left(\frac{dk}{d\omega}\right)_{\omega_0} = \frac{1}{v_G}$

A good practise is to introduce the change of time⁶ $T = t - z/v_G$. With such a change the observer is travelling at the same speed as the pulse. The equation (24) then becomes simply

$$i \frac{\partial E}{\partial z} = +\frac{1}{2}k_2 \frac{\partial^2 E}{\partial T^2} \quad (25)$$

0.3.2 Propagation in a purely dispersive material

To solve the equation of propagation (eq. (25)) we use the so-called *Fourier-transform method*, which is based on the definition of the pulse from its spectrum

$$U(z, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{U}(z, \omega) e^{-i\omega T} d\omega \quad (26)$$

and allows transforming the eq. (25) in the frequency domain by using

$$\mathcal{F}[\partial_z U(z, T)] = \partial_z \tilde{U}(z, T) \quad (27a)$$

$$\mathcal{F}[(\partial_T)^n U(z, T)] = (-i\omega)^n \tilde{U}(z, T) \quad (27b)$$

The equation (25) is then

$$i \frac{\partial \tilde{U}(z, \omega)}{\partial z} = -\frac{1}{2}k_2 \omega^2 \tilde{U}(z, \omega) \quad (28)$$

which is readily solved as

$$\tilde{U}(z, \omega) = \tilde{U}(0, \omega) \exp\left(\frac{i}{2}k_2 \omega^2 z\right) \quad (29)$$

where $\tilde{U}(0, \omega)$ is the spectrum of the pulse at $z = 0$. It is clear from the eq. (29) that the spectrum in the initial pulse is **not modified** during the propagation. As this equation suggests each spectral component ω acquires a different phase during the propagation. The temporal shape of the pulse after a distance z is then

$$U(z, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(0, \omega) \exp\left(\frac{i}{2}k_2 \omega^2 z - i\omega T\right) d\omega \quad (30)$$

where $\tilde{U}(0, \omega)$ is the inverse Fourier transform of the pulse at $z = 0$

$$\tilde{U}(0, \omega) = \int_{-\infty}^{+\infty} U(0, T) e^{i\omega T} dT \quad (31)$$

0.3.3 Propagation of a Gaussian pulse

As an example we consider here that we have a Gaussian pulse propagating in a dispersive medium. Its envelope is given by

$$U(0, T) = \exp\left(-\frac{T^2}{2 T_0^2}\right) \quad (32)$$

⁶Note that any function $F(T = t - z/v_G)$ is a solution of the equation $(\partial_z + k_1 \partial_t) F = 0$ since:

$$(\partial_z + k_1 \partial_t) F(T) = \frac{\partial F}{\partial T} \frac{\partial T}{\partial z} + k_1 \frac{\partial F}{\partial T} \frac{\partial T}{\partial t} = \frac{\partial F}{\partial T} = -\frac{1}{v_G} \frac{\partial F}{\partial T} + k_1 \frac{\partial F}{\partial T} = 0$$

With this new variable $\partial_t = \partial_T$.

where T_0 is the half-width (at $1/e$ -intensity point). Although this definition makes the calculations easier but it is in practice more convenient to work with the full-width at half-maximum (FWHM) instead of T_0

$$T_{\text{FWHM}} = 2\sqrt{\ln 2} T_0 \approx 1.665 T_0 \quad (33)$$

By inserting the eq. (32) into eq. (30) and calculating the integration we obtain the shape of the pulse at a distance z :

$$U(z, T) = \frac{T_0}{\sqrt{T_0^2 - ik_2z}} \exp \left[-\frac{T^2}{2(T_0^2 - ik_2z)} \right] \quad (34)$$

which is still a Gaussian pulse. However the amplitude and the duration of the pulse are modified. In particular the exponential can be written as

$$\begin{aligned} \exp \left[\frac{T^2}{2(T_0^2 - ik_2z)} \right] &= \exp \left[-\frac{T^2 (t_0^2 + ik_2z)}{2(T_0^4 + k_2^2z^2)} \right] \\ &= \exp \left[-\frac{T_0^2 T^2}{2(T_0^4 + k_2^2z^2)} \right] \exp \left[-i \frac{k_2z T^2}{2(T_0^4 + k_2^2z^2)} \right] \end{aligned} \quad (35)$$

From the first exponential we find that the pulse is actually getting broader. Its duration is now given by

$$T^2(z) = \frac{T_0^4 + k_2^2z^2}{T_0^2} \rightarrow \boxed{T(z) = T_0 \sqrt{1 + \left(\frac{z}{L_D} \right)^2}} \quad (36)$$

where the *dispersion length* $L_D = T_0^2/|k_2|$. For distance $L > L_D$ the effect of dispersion can be observed. It is clear that the shorter the pulse the shorter the dispersion length. Therefore, for a given physical length of the dispersion element, short pulses will be more affected by dispersion than long pulses. In the context of laser cavity, this means that the effect of dispersion will be more severe when the targeted pulses are short.

Obviously the appearance of complex number in the expression of $U(z, T)$ in both the amplitude and the exponential part (eq. (34)) implies that the phase is modified as the pulse propagates. We can rewrite $U(z, T)$ as

$$U(z, T) = |U(z, T)| \exp [i\phi(z, T)] \quad (37)$$

where the phase $\phi(z, T)$ is then given by

$$\phi(z, T) = \frac{-\text{sgn}(k_2) (z/L_D)}{1 + (z/L_D)^2} \frac{T^2}{2T_0^2} + \frac{1}{2} \tan^{-1} \left(\frac{z}{L_D} \right) \quad (38)$$

Since the phase depends on time the instantaneous frequency will differ at different location in the pulse

0.3.4 Time-bandwidth product

In the previous calculation we only looked at the envelope of the pulse. Because of the shift properties of the Fourier transform adding the carrier phase $\exp(i\omega_0 t)$ would simply

result in a shift in the Fourier transform and the spectrum would then be centered around ω_0 . Let's assume now that the pulse is described by the more general complex envelope

$$E(t) = \exp\left(-\frac{t^2}{2T_0^2}\right) \exp\left[-i\left(\omega_0 t + \frac{1}{2}bt^2\right)\right] \quad (39)$$

where b is a constant. The time-varying phase shift of the sinusoidal signal (carrier) underneath the Gaussian envelope is $\phi(t) = \omega_0 t + 1/2 bt^2$. The instantaneous frequency ω_i is the rate at which the total phase of sinusoidal carrier signal rotates and is therefore defined as

$$\omega_i(t) = \frac{d\phi(t)}{dt} \quad (40)$$

In the particular case of eq. (39), the instantaneous frequency is

$$\omega_i(t) = \omega_0 + bt \quad (41)$$

is linearly drifting along the pulse itself (Fig. 3) : this is the *chirp*.

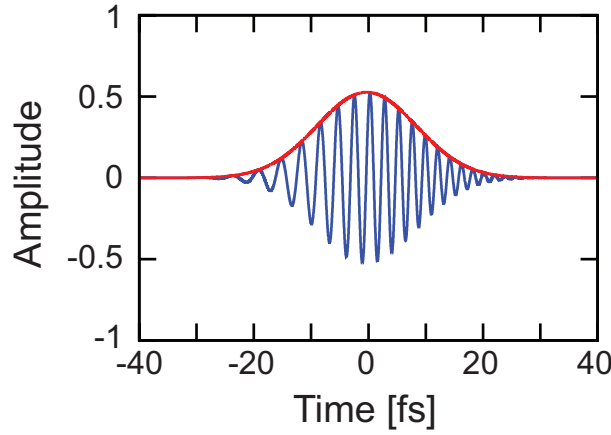


Figure 3: chirped pulse

To calculate the spectrum of the field $E(t)$ (eq. (39)) can be written as

$$E(t) = \exp\left(-\frac{\Gamma t^2}{2}\right) \exp(-i\omega_0 t) \quad (42)$$

where $\Gamma = 1/T_0^2 + ib$. The Fourier transform of eq. (42) is

$$\begin{aligned} \tilde{E}(\omega) &= \frac{\sqrt{2\pi}}{\sqrt{\Gamma}} \exp\left(\frac{-(\omega - \omega_0)^2}{2\Gamma}\right) = T_0 \frac{\sqrt{2\pi}}{\sqrt{1 + ibT_0^2}} \exp\left[\frac{-(\omega - \omega_0)^2 T_0^2}{2(1 + ibT_0^2)}\right] \\ &\propto \exp\left[\frac{-(\omega - \omega_0)^2 T_0^2 (1 - ibT_0^2)}{2(1 + b^2 T_0^4)}\right] \end{aligned} \quad (43)$$

As previously the spectrum has still a Gaussian shape. Looking at the bandwidth of this Gaussian spectrum⁷ and using eq. (33) we can write

$$\exp\left[\frac{-(\omega - \omega_0)^2 T_0^2}{2(1 + b^2 T_0^4)}\right] = \exp\left[-4 \ln 2 \frac{(\Delta\omega)^2}{(\Delta\omega_{\text{FWHM}})^2}\right] \quad (44)$$

⁷This means that we are not considering the imaginary part of the argument of the exponential, since this is a phase term.

where $\Delta\omega = \omega - \omega_0$ and $\Delta\omega_{\text{FWHM}}$ is the bandwidth of the spectrum. This yields

$$\begin{aligned} \frac{(\Delta\omega_{\text{FWHM}})^2}{4 \ln 2} &= \frac{1 + b^2 T_0^4}{T_0^2} \\ \Leftrightarrow (\Delta\omega_{\text{FWHM}})^2 &= 4\pi^2 (\Delta\nu_{\text{FWHM}})^2 = 4 \ln 2 \left(\frac{1 + b^2 T_0^4}{T_0^2} \right) \\ \Leftrightarrow \Delta\nu_{\text{FWHM}} T_0 &= \Delta\nu_{\text{FWHM}} \frac{T_{\text{FWHM}}}{2\sqrt{\ln 2}} = \frac{\sqrt{\ln 2}}{\pi} (1 + b^2 T_0^4) \\ \implies \Delta\nu_{\text{FWHM}} T_{\text{FWHM}} &= \frac{2 \ln 2}{\pi} (1 + b^2 T_0^4) \end{aligned} \quad (45)$$

$$\implies \boxed{\Delta\nu_{\text{FWHM}} T_{\text{FWHM}} = \frac{2 \ln 2}{\pi} (1 + b^2 T_0^4)} \quad (46)$$

where $\Delta\nu_{\text{FWHM}}$ and T_{FWHM} are respectively the full-width half maximum of the spectrum (in Hz) and of the pulse duration in second. This is called the *time-bandwidth* product. For a Gaussian pulse without any chirp this time-bandwidth product is simply

$$\Delta\nu_{\text{FWHM}} T_{\text{FWHM}} = \frac{2 \ln 2}{\pi} \simeq 0.441 \quad (47)$$

by measuring the time-bandwidth product it is possible to evaluate the quality of the pulse and how close the pulse is from *transform-limited*.

0.4 The necessity of balancing of the dispersion

0.4.1 Relative pulse broadening due to dispersion

When a pulse⁸ propagated through a dispersive material, it broadens. The relative pulse broadening is given by⁹

$$\begin{aligned} \left(\frac{\delta T_0}{T_0} \right)_D &= \frac{T_0(L) - T_0}{T_0} = \frac{T_0 \sqrt{1 - (L/L_D)^2} - 1}{T_0} \\ &\simeq \frac{1}{2} \left(\frac{L}{L_D} \right)^2 = \frac{1}{2} \left(\frac{k_2 L}{T_0^2} \right)^2 \\ \implies \left(\frac{\delta T_0}{T_0} \right)_D &\simeq \frac{1}{2} \left[\frac{\phi''(\omega_0)}{T_0^2} \right]^2 \end{aligned} \quad (48)$$

And using T_{FWHM} instead of T_0 (eq. (33)):

$$\left(\frac{\delta T_{\text{FWHM}}}{T_{\text{FWHM}}} \right)_D = 8 (\ln 2)^2 \left[\frac{\phi''(\omega_0)}{T_{\text{FWHM}}^2} \right]^2 \quad (49)$$

Note that we only considered the second order dispersion k_2 . For pulses shorter than 30 fs, this is no longer valid and the third order (k_3) dispersion will need to be included.

⁸As previously, we take a Gaussian pulse for simplicity of the calculations.

⁹The subscript D indicates that this broadening is due to the propagation through a dispersive material.

0.4.2 Relative pulse broadening due to the gain medium

The gain medium has a Lorentzian shape given by

$$G(\omega) \simeq \frac{g_0}{1 + j \frac{\omega_0}{\Delta\omega}} \simeq g_0 \left(1 - j \frac{\omega_0}{\Delta\omega} - \frac{\omega_0^2}{\Delta\omega^2} \right) \quad (50)$$

where g_0 is the gain at the central frequency ω_0 . As we did before, we could express this equation in the time domain yielding a term $\left(d^2/dt^2 \right)$ which is very similar as the one introduced by the linear dispersion of the material eq. (25)). We then see that the gain itself leads to dispersion of the pulse traveling in the laser cavity. This can be seen as the result of the limited time-response (temporal aspect of the spectral bandwidth) of the gain medium. The influence of the gain medium on the pulse duration can be evaluated and is given by

$$T_{\text{FWHM}}^2(z) = T_{\text{FWHM},0}^2 + \frac{(8 \ln 2) \ln G_0}{\Delta\omega_a^2} \quad (51)$$

where $G_0 = \exp(g_0 z)$. For a small change of the pulse duration this equation becomes¹⁰

$$\left(\frac{\delta T_{\text{FWHM}}}{T_{\text{FWHM}}} \right)_G = \frac{4 \ln 2 \ln G_0}{T_{\text{FWHM},0}^2 \Delta\omega_a^2} \quad (52)$$

0.4.3 Limit of the pulse duration

To estimate the total pulse broadening it is important to consider both the influence of the gain material and the one from the cavity itself. The relative pulse change is then

$$\left(\frac{\delta T_{\text{FWHM}}}{T_{\text{FWHM}}} \right)_{\text{total}} = 8 (\ln 2)^2 \left[\frac{\phi''(\omega_0)}{T_{\text{FWHM}}^2} \right]^2 + \frac{4 \ln 2 \ln G_0}{T_{\text{FWHM},0}^2 \Delta\omega_a^2} \quad (53)$$

The two contributions scale differently with respect to the pulse duration. The influence of the dispersion is inversely proportional to T_{FWHM}^4 whilst the influence of the gain is inversely proportional to T_{FWHM}^2 . The shorter the pulses the more severe is the influence of dispersion. To estimate the pulse width at which the GDD starts to be relevant, we equate the two contributions and obtain

$$T_{\text{FWHM}} = \sqrt{\frac{2 \ln 2}{\ln G_0}} |\phi''(\omega_0)| \Delta\omega_a = \sqrt{\frac{2 \ln 2}{g_0 z}} |\phi''(\omega_0)| \Delta\omega_a \quad (54)$$

As an example let's consider a Ti:Sa laser ($\Delta\omega_a \simeq 2\pi \times 100$ THz), $\phi''(\omega_0) = 100$ fs² (this corresponds to ~ 2 mm of quartz glass in the cavity, and $g_0 z = 0.1$). Using Eq. (54), we obtain that GDD becomes relevant for pulse with pulse duration larger than $T_{\text{FWHM}} = 162$ fs. Considering the full available bandwidth of the gain material, we could expect pulses as shorter as $1/\Delta\omega_a \simeq 4$ fs, however it is essential to balance the effect of GDD to achieve such extreme short pulse duration. In fact, when GDD is perfectly balanced third order dispersion (TOD) will start influencing significantly the pulse duration and will need to be addressed as well. This becomes relevant for pulses shorter than 30 fs.

¹⁰We remind that $\sqrt{1+x} \underset{x \rightarrow 0}{\simeq} 1 + \frac{1}{2} x$

0.5 Balancing the dispersion

From the definition of the group delay (eq. (11)) we see that looking at the influence of the dispersion on the pulse duration can be seen as the modification of the group delay, which can be simply written as

$$\tau_G = \phi'(\omega_0) + \phi''(\omega - \omega_0) + \frac{1}{2}\phi'''(\omega - \omega_0)^2 + \dots \quad (55)$$

Therefore it is clear that limiting the effect of dispersion require to limit the effects of ϕ'' and ϕ''' . As we saw previously, the different elements of the laser cavity (mirrors, gain material... etc) lead to temporal broadening of the pulse (supposing that we have managed to generate a pulse in first place) because of positive dispersion, also called *normal dispersion*. In this regime of dispersion red component are faster than blue ones. Balancing the dispersion requires therefore to “create” an element which induces “negative dispersion”. For any transparent material, the refractive index (and therefore the chromatic dispersion) is a monotonic function of the wavelength. One way to obtain negative dispersion could be to place an absorbing element in the cavity. Close to a resonance the dispersion will reverse. Obviously this is not the best strategy since inserting an absorbent in the cavity will strongly increase the lasing threshold and degrade the performance of the laser. It is therefore important to use individual optical elements, which will not induce extra losses while still inducing a negative dispersion.

0.5.1 Four prisms method

In 1984, R.L. Fork, O.E. Martinez and J.P. Gordon published a very important work for ultrashort pulse generation in the context of dispersion management¹¹. It is based on the use of prisms. By contrast with the use of gratings the cost of fabrication of prism is very low and the transmission loss can be very low, especially when the prisms are Brewster-cut. It is well known that when a poly-chromatic beam arrives on a prism it is dispersed so that the blue component of the spectrum are more deviated than the red ones. Placing a second prism upside-down compared the first one, the red component will have to travel through a larger piece of glass. As a consequence the *optical path* for the red component becomes larger than for the blue component. In other words the pair of prism has induced **negative dispersion**. Of course at the end of the pair of prism, the beam is spatially broader and a second pair of prisms should be used to refocus the beam spatially.

The phase induced but the first prism is given by

$$\phi = |k| \ell_p \cos \beta(\omega) \quad (56)$$

where ℓ_p is the distance between the apexes of the prisms and $\beta(\omega)$ is the angle between the diffracted ray (at a frequency ω) and the line joining the prisms' apexes. Following the work from R.L. Fork and co-workers it is convenient to introduce the optical path

$$P = n_{\text{air}} \ell_p \cos \beta = \ell_p \cos \beta \quad (57)$$

The second order derivative of the phase is then given by

$$\phi''(\omega) = \frac{\lambda^3}{2\pi c^2} \frac{d^2 P}{d\lambda^2} \quad (58)$$

¹¹“Negative dispersion using pairs of prisms,” Opt. Lett. (9) p.150–152 (1984)

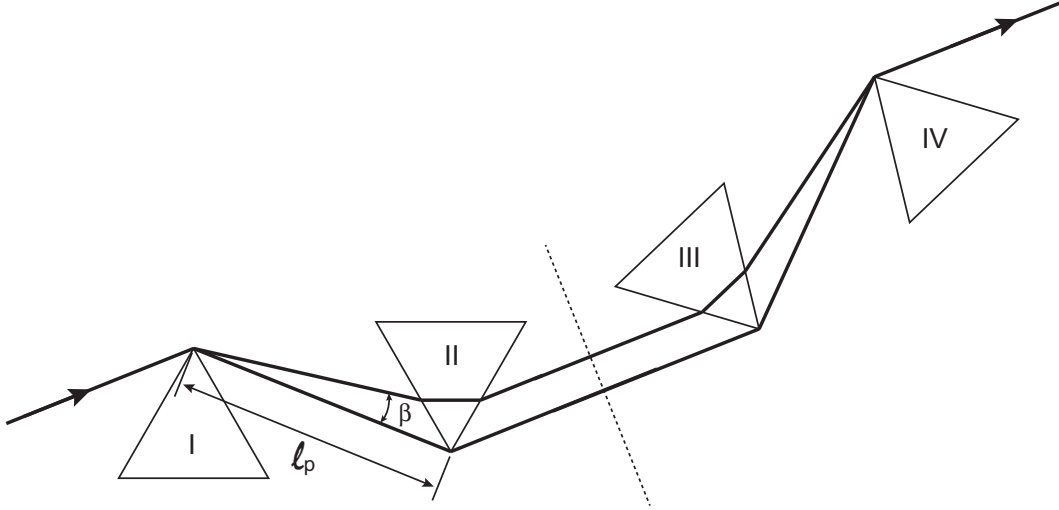


Figure 4: Four-prisms sequence for negative dispersion. The dashed line represents the axis of symmetry of the system. [R.L. Fork, O.E. Martinez and J.P. Gordon, Opt. Lett. (9) 150 (1984)]

Since $P = P(\beta)$ and the diffraction angle is $\beta = \beta[n(\omega)]$, where $n(\omega)$ is the refractive index of the prism we need to use the chain rule for derivation

$$\frac{dP}{d\lambda} = \frac{dP}{d\beta} \frac{d\beta}{dn} \frac{dn}{d\lambda} \quad (59)$$

After calculations it comes

$$\frac{d^2P}{d\lambda^2} = 2\ell_p \left\{ \left[\frac{d^2n}{d\lambda^2} + \left(2n - \frac{1}{n^3} \right) \left(\frac{dn}{d\lambda} \right)^2 \right] \sin \beta - 2 \left(\frac{dn}{d\lambda} \right)^2 \cos \beta \right\} \quad (60)$$

Moreover since β is of the order of the angular deviation of the ray bundle $\sin \beta \ll \cos \beta$. Therefore the phase induced by the **first prism** due to the angular dispersion is

$$\phi''_{angular}(\omega_0) \simeq -4\ell_p \frac{\lambda_0^3}{2\pi c^2} \left(\frac{dn}{d\lambda} \right)^2 \quad (61)$$

Of course we need to add the contribution of the second prism, which contributes positively since the beam is traveling inside the second prism. The propagation inside the second prism yields a phase shift $\phi = |k| \ell_{inside}$. In other words $P_{inside} = n(\lambda) \ell_{inside}$ and therefore the contribution of the second prism to the dispersion is

$$\phi''_{in.}(\omega_0) = \ell_{in.} \frac{\lambda_0^3}{2\pi c^2} \left(\frac{d^2n}{d\lambda^2} \right) \quad (62)$$

The total dispersion induced by the first pair of prisms is then

$$\boxed{\phi(\omega) = -4\ell_p \frac{\lambda_0^3}{2\pi c^2} \left[\left(\frac{dn}{d\lambda} \right)_{\lambda=\lambda_0} \right]^2 + \ell_{in.} \frac{\lambda_0^3}{2\pi c^2} \left(\frac{d^2n}{d\lambda^2} \right)_{\lambda=\lambda_0}} \quad (63)$$

It is clear that the use of prisms can induced a negative dispersion (the first term is negative) and the total amount of dispersion can be finely tuned with the amount of glass

introduced by the second prism. As said previously after the pair of prisms the beam is spatially broader and it is necessary to use a second pair of prisms to refocus the beam. The easiest is to simply place a flat mirror after the first pair of prisms (indicated by the dashed line on Fig. 4).