Nonlinear Optics
Nonlinear propoagation of optical pulses

5.1 Nonlinear propagation of optical pulses

5.1.1 Plane-wave decomposition

Historically the study of the propagation of pulses in optical fibres is motivated by telecommunications and by the theory of interferometers. Since a monochromatic wave $E = \mathcal{E}e^{-i(\omega t-kz)}$ cannot carry any information, it is important to modulated the amplitude $\mathcal{E}$ of the wave. The faster we can modulate, the more information we can encode onto the wave. This is directly linked with the spectral bandwidth of the input field through Fourier theory such that this input field is described by

$$E(t) = \mathcal{F}\left[E(\omega)\right](t) = \int_{-\infty}^{+\infty} \widetilde{E}(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} \tag{5.1}$$

where the optical spectrum[1] is expressed by the inverse Fourier transform

$$\widetilde{E}(\omega) = \mathcal{F}^{-1}\left[E(t)\right](\omega) = \int_{-\infty}^{+\infty} E(t)e^{i\omega t}dt \tag{5.2}$$

Of course, in its most general form the electric field is a vectorial quantity which depends on both time and space:

$$\mathbf{E}(\mathbf{r},t) = \int \widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\frac{d\omega}{2\pi} \tag{5.3}$$

or by using a plane-wave decomposition:

$$\mathbf{E}(\mathbf{r},t) = \int \widetilde{\mathbf{E}}(\mathbf{k},\omega)e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3} \tag{5.4}$$

where $\mathbf{k}$ is the wavevector. Its norm is the wavenumber $k = ({n\omega}/{c})$ and n is a function of the frequency ω.

5.1.2 The influence of dispersion

Group velocity

Let us consider an optical pulse propagating along the $z-$direction. Then from eq. (5.4) has a simpler form=

$$E(z,t) = \int_{-\infty}^{+\infty} \widetilde{E}(z,\omega)e^{-i(\omega t-kz)}\frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} \widetilde{A}(\omega-\omega_0)e^{-i(\omega t-kz)}d\omega \tag{5.5}$$

where $A(\omega-\omega_0)$ is the complex amplitude of each spectral component. In this integral the wave-number k is also a function of the angular frequency $k = k(\omega-\omega_0)$. And the

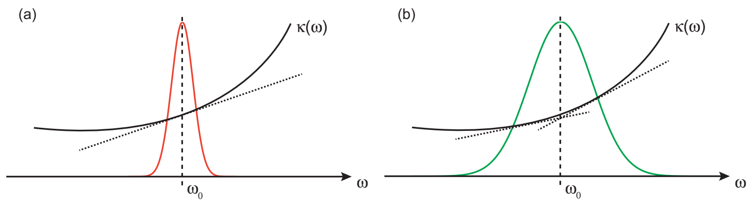

Figure 5.1: Relationship between the spectrum of the pulse and the normalised dispersion relation $k(\omega)$ in the case of (a) long and (b) short pulses.

dispersion properties of the material where the wave is propagating is described by the dispersion relation $k = k(\omega)$.

Depending on the initial spectral bandwidth of the pulse this function can be Taylor-expanded around the central frequency of the pulse ω_0 in order to be taken into account (Fig. 5.1). Let's assume that, over the bandwidth of the pulse we can truncate the Taylor expansion at its first order:

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0} (\omega - \omega_0) + o(\omega^2) \tag{5.6}$$

where $k_0 = k(\omega_0)$ and the subscript in the derivative indicates that the derivative is calculated at $\omega = \omega_0$. Inserting this relation of dispersion into eq. (5.5) yields

$$\begin{aligned} E(z,t) &= \int_{-\infty}^{\infty} \widetilde{A}(\omega - \omega_0)\ e^{-i\left\{\omega t - \left[k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0)\right]z\right\}} d\omega \\ &= \int_{-\infty}^{\infty} \widetilde{A}(\omega - \omega_0)\ e^{-i\left\{(\omega-\omega_0+\omega_0)t - \left[k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0)\right]z\right\}} d\omega \\ &= e^{-i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \widetilde{A}(\omega - \omega_0)\ e^{-i\left\{(\omega-\omega_0)t - \left[\left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0)\right]z\right\}} d\omega \\ E(z,t) &= e^{-i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \widetilde{A}(\Omega)\ e^{-i\left\{\Omega t - \left[\left(\frac{dk}{d\omega}\right)_{\omega_0}\Omega\right]z\right\}} d\Omega \end{aligned} \tag{5.7}$$

with $\Omega = \omega - \omega_0$. Note that this integral is simply the Fourier transform[2] of the spectral amplitude $A(\Omega)$:

$$E(z,t) = A\left[t - \left(\frac{z}{v_G}\right)\right] e^{-i(\omega_0 t - k_0 z)} \tag{5.8}$$

where we have defined the group velocity[3] v_G by

$$\frac{1}{v_G} = \left(\frac{dk}{d\omega}\right)_{\omega_0} \tag{5.9}$$

[1]Note that the spectrum is a complex quantity for which we can define an amplitude $\left|\widetilde{E}(\omega)\right|$ and a spectral phase $\varphi(\omega)$.

[2]We remind that the Fourier transform is defined as

$$G(\omega) = \mathcal{F}\left[g(t)\right] = \int_{\infty}^{\infty} g(t) e^{i\omega t} d\omega \ \text{ and } \ \mathrm{g(t)} = \frac{1}{2\pi}\int_{\infty}^{\infty} \mathrm{G}(\omega)\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}\mathrm{d}\omega$$

We also remind the shift properties of the Fourier transform:

$$\mathcal{F}\left[g(t-\tau)\right] = e^{t\omega\tau}\mathcal{F}\left[g(t)\right] = e^{t\omega\tau}G(\omega)$$

[3]The group velocity is in general different from the phase velocity.

The group velocity can be interpreted as the speed at which the wave-packet propagates. From the relation of dispersion this is also obvious that this the inverse of the slope of this relation. As we see for the eq. (5.8), the pulse does not change its shape but acquire a delay as it propagates along z. After a distance L in the disperse medium the pulse has accumulated a delay

$$\tau_G = \frac{L}{v_G} \tag{5.10}$$

and a phase-shift $\phi(\Omega) = L\ k(\Omega)$. From the definition of the group velocity (eq. (5.9) we see that the *group delay* is defined as

$$\tau_G = \phi'(\Omega) \tag{5.11}$$

Group velocity dispersion

In the previous section, we truncated the Taylor expansion at the first order. This corresponds to the dashed line on Fig. 5.1. Although this is perfectly valid for the red spectrum on that figure, it is no longer the case for the larger spectrum (green on the figure), corresponding to a pulse with a shorter duration[4].

Since the group velocity is defined at the inverse of the slope of the relation of dispersion (dashed line on Fig. 5.1), it is clear that in the case of the short-duration pulse (broad spectrum) the group velocity that is calculated at the edge of the spectrum are not identical since the slopes at different from each others. In others words, as the pulse propagates different part of its spectrum travel at different group velocity, yield a *dispersion of the group-delay* $\Delta\tau_G$. This can be evaluated from the fastest (with frequency ω_1) and the slowest (with a frequency ω_2) spectral component:

$$\Delta\tau_G = L\left(\frac{1}{v_G^{(2)}} - \frac{1}{v_G^{(1)}}\right) = L\left[\left(\frac{dk}{d\omega}\right)_{\omega_2} - \left(\frac{dk}{d\omega}\right)_{\omega_1}\right] \tag{5.12}$$

If we consider that the group-velocity $v_G^{(1)}$ is not too different from $v_G^{(2)}$ then we can use a Taylor expansion to re-calculate $\Delta\tau_G$

$$\Delta\tau_G = L\left[\frac{1}{v_G(\omega_2)} - \frac{1}{v_G(\omega_1)}\right] = L\left[\frac{1}{v_G(\omega_1)} + \frac{d}{d\omega}\left(\frac{1}{v_g}\right)_{\omega_1}(\omega_2 - \omega_1) - \frac{1}{v_G(\omega_1)}\right] \tag{5.13}$$

And since

$$\frac{d}{d\omega}\left(\frac{1}{v_G}\right) = \frac{d}{d\omega}\left(\frac{dk}{d\omega}\right) = \frac{d^2k}{d\omega^2} \tag{5.14}$$

we can rewrite broadening $\Delta\tau_G$ as

$$\Delta\tau_G = L\underbrace{\left|\left(\frac{d^2k}{d\omega^2}\right)_{\omega_0}\right|}_{\text{GVD}}(\omega - \omega_0) \tag{5.15}$$

As previously we can use the derivative of the phase and write

$$\Delta\tau_G = \phi''(\omega_0)(\omega - \omega_0) \tag{5.16}$$

[4] The curve corresponding to $k(\omega) = n(\omega)\omega/c$ does not vary very much. We plotted here the normalised dispersion relation $\kappa(\omega) = k(\omega) - \left(\omega k(\omega_{\max})/\omega_{\max}\right)$. Regarding the slope (dashed line on the Fig. 5.1), this only adds a constant value to the derivative of $k(\omega)$ with respect to ω and therefore does not affect the quantitative description of the dispersion since this constant will vanishes in eq. (5.12).

where $\phi''(\omega_0)$ is often called the *group-delay dispersion* (GDD). Considering eq. (5.11) we see that we are actually doing a Taylor expansion of the equation for the group delay. Including higher-order term this is

$$\tau_G = \phi'(\omega_0) + \phi''(\omega - \omega_0) + \frac{1}{2}\underbrace{\phi'''(\omega - \omega_0)^2}_{TOD} + o(\omega^3) \tag{5.17}$$

where TOD is the influence of *third order dispersion.* Regarding the units of the different variable that we just introduced. We see from eq. (5.15) that the *group velocity dispersion* (GVD) has the unit of [time]^2/[length] whilst the group delay dispersion has the unit of [time]^2. In practise the GDD is mostly used to characterise the dispersion of mirror, where the parameter L does not have much sense. It is usually measured in fs^2.

It is interesting to point that the impossibility to use the linear expansion of the dispersion relation yields the introduction of the second derivative $\left(d^2k/d\omega^2\right)$. It was actually clear from the beginning that our initial Taylor expansion was no longer valid and we had to introduce the next term in the definition of $k(\omega)$:

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega - \omega_0) + \frac{1}{2}\left(\frac{d^2k}{d\omega^2}\right)_{\omega_0}(\omega - \omega_0)^2 + o(\omega^3) \tag{5.18}$$

It is now obvious to associate the second order term in this expansion to the GVD.

5.1.3 The nonlinear Schrodinger equation

We already saw that in the presence of the $\chi^{(3)}$ material the refractive index is modified such as it can expressed as[5]

$$n = n_0 + \overline{n_2}\,|E|^2 \tag{5.19}$$

Since the wavenumber is directly linked with the refractive index, we can write

$$k = \frac{\omega n}{c} = \frac{\omega n_0(\omega)}{c} + \frac{\omega\overline{n_2}}{c}|E|^2 = k\left(\omega, |E|^2\right) \tag{5.20}$$

The most convenient way to derive a global equation which would describe the evolution of the envelope of the pulse taking into account both linear (dispersion) and nonlinear (Kerr) effect is to Taylor expand the wavenumber:

$$k\left(\omega, |E|^2\right) = k_0 + \underbrace{\left(\frac{\partial k}{\partial \omega}\right)}_{k_1}(\omega - \omega_0) + \frac{1}{2}\underbrace{\left(\frac{\partial^2 k}{\partial \omega^2}\right)}_{k_2}(\omega - \omega_0)^2 + \underbrace{\left(\frac{\partial k}{\partial |E|^2}\right)}_{\frac{\omega\overline{n_2}}{c}}|E|^2 \tag{5.21}$$

Finally we can use the properties of the Fourier transform by considering that a change $\Delta\omega = \omega - \omega_0$ yields a change of wavenumber $\Delta k = k - k_0$ so that from the expression of the electric field in the space-time domain and its reciprocal $\{k - \omega\}$ domain:

$$\widetilde{E}(\Delta k, \Delta\omega) \propto \iint E(z,t)e^{+i(\Delta\omega t - \Delta k z)}dzdt \tag{5.22a}$$

$$E(z,t) \propto \iint \widetilde{E}(\Delta k, \Delta\omega)e^{-i(\Delta\omega t - \Delta k z)}d\omega dk \tag{5.22b}$$

[5]Note that previously we introduced n_2, but used the expression $n = n_0 + n_2 I$. Remember that the relation between the intensity I and the square of the amplitude $|E|^2$ contains several constants which are here included in $\overline{n_2}$.

we can identify that

$$\Delta k = (k - k_0) \longleftrightarrow -i\partial_z \tag{5.23a}$$

$$\Delta\omega = (\omega - \omega_0) \longleftrightarrow i\partial_t \tag{5.23b}$$

The operator equation (eq. (5.21) applied to the envelope of the input field is straightforwardly given by

$$-i\frac{\partial E}{\partial z} = ik_1\frac{\partial E}{\partial t} - \frac{1}{2}k_2\frac{\partial^2 E}{\partial t^2} + \frac{\omega\overline{n_2}}{c}|E|^2 E \tag{5.24}$$

Note that for low amplitudes, we can neglect the Kerr-effect. And in the case where there is not second order dispersion this simply becomes

$$-i\left(\frac{\partial}{\partial_z} + k_1\frac{\partial}{\partial z}\right)E = 0 \tag{5.25}$$

which has a trivial solution that can be expressed by any function of the variable $(z - {}^t/_{k_1})$, corresponding to the situation when the pulse propagates at the group velocity without changing its shape. A good practise is to introduce the change of variable $T = t - {}^z/_{v_g}$, which means that the observer is travelling at the same velocity of the pulse. The eq. (5.24) then becomes

$$\boxed{i\frac{\partial E}{\partial z} = \frac{1}{2}k_2\frac{\partial^2 E}{\partial T^2} - \frac{\omega\overline{n_2}}{c}|E|^2 E} \tag{5.26}$$

By analogy with the quantum mechanics, this equation is called the *nonlinear Schrodinger equation*. We remind that in quantum mechanics the Schrodinger equation for one particle moving in one dimension is

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{5.27}$$

where H the Hamiltonian of the system consists of the sum of a kinetic $\hat{T}$ and a potential $\hat{V}$ energy. Expressing the kinetic energy term yields the direct correspondence between the Schrodinger equation that we know from quantum mechanics and the eq. (5.26):

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \tag{5.28}$$

where the kinetic energy operator in quantum mechanics corresponds to the dispersion operator in nonlinear optics and the potential energy, which is linear for a single particle $V(x)\psi$ corresponds to a nonlinear term proportional to $|E|^2 E$ in the present case[6]. Numerically the integration of the nonlinear Schrodinger equation will use very similar technique as the one used in quantum mechanics.

5.2 Pulse propagation in optical fibre

5.2.1 Step-index fibre

A step-index fibre is schematically represented on Fig. 5.2. For such fibres the refractive index of the core region n_{co} is supposed to be constant and so is the refractive of the cladding n_{cl}. To allow guidance in the core the refractive index in the core has to be larger

[6]Of course the coordinate have the following correspondence $z_{\mathrm{NLO}} \leftrightarrow t_{\mathrm{QM}}$ and $t_{\mathrm{NLO}} \leftrightarrow x_{\mathrm{QM}}$.

than the cladding one. For telecommunication fibres are made of silica and the refractive index is modified by doping either the core with Germanium, in order to increase its refractive index ($n_{co} > n_{cl}$) or the cladding with Fluorine or potassium in order to lower its refractive index ($n_{cl} < n_{co}$). The conventional fibre for telecommunication has a core

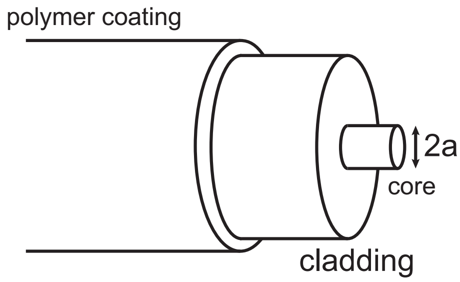

Figure 5.2: step-index fibre. The core has a radius a and a refractive index n_{co}. The cladding has a refractive index n_{cl}. The polymer coating acts as a protection for the fibre as well as a way to eliminate the light leaking out of the core.

diameter of $\sim 8.2\mu$m and an external diameter of 125μm. The *profile height parameter*

$$\Delta n = \frac{1}{2}\left(1 - \frac{n_{co}^2}{n_{cl}^2}\right) \simeq \frac{n_{co} - n_{cl}}{n_{co}} \simeq 5 \times 10^{-3} \tag{5.29}$$

Although this is a small value it is enough to guide light very efficiently. The best transmission loss figures so far is 0.154 dB/km.

At a given frequency the optical fibre can only support on finite number of optical guided modes. Separating the variables we can express the electric field as

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = F(x, y)\tilde{A}(z, \omega - \omega_0)e^{i\beta_0 z} \tag{5.30}$$

where $F(x, y)$ is the transverse distribution of the mode and $\widetilde{A}(z, \omega - \omega_0)$ the amplitude spectrum, which is slowly varying during propagation. And the electric field has to be a solution of the wave equation

$$\nabla^2 \tilde{\mathbf{E}} + \epsilon(\omega)k_0^2\, \tilde{\mathbf{E}} = 0 \tag{5.31}$$

Because of the way we expressed the electric field (eq. (5.30)), it is convenient to express the Laplacian operator as $\nabla^2 = \partial_{zz} + \triangle_\perp^2$, where $\triangle_\perp^2$ is the transverse Laplacian operator. Since the longitudinal part gives

$$\partial_{zz}\left[F(x, y)\widetilde{A}(\omega - \omega_0)e^{i\beta_0 z}\right] = F(x, y)\left[\left(\partial_{zz}\widetilde{A}\right)e^{i\beta_0 z} + 2i\beta_0\left(\partial_z\widetilde{A}\right)e^{i\beta_0 z} - \beta_0^2\widetilde{A}e^{i\beta_0 z}\right]$$

the Helmholtz equation becomes

$$F(x, y)\left[\cancel{\left(\partial_{zz}\widetilde{A}\right)e^{i\beta_0 z}} + 2i\beta_0\left(\partial_z\widetilde{A}\right)e^{i\beta_0 z} - \beta_0^2\widetilde{A}e^{i\beta_0 z}\right] + \left[\triangle_\perp F(x, y)\right]\widetilde{A}e^{i\beta_0 z} + \epsilon(\omega)k_0^2\widetilde{A}e^{i\beta_0 z} = 0$$

where the first term is neglected in the slowly varying envelope approximation (SVEA). We can rewrite in the following form

$$\frac{1}{\widetilde{A}}\left[2i\beta_0\left(\partial_z\widetilde{A}\right) - \beta_0^2\widetilde{A}\right] = -\frac{1}{F(x, y)}\left[\triangle_\perp F(x, y) + \epsilon(\omega)k_0^2 F(x, y)\right] \tag{5.32}$$

where we see that $\widetilde{A}$ (resp. $F(x, y)$) only appears on the left (resp. right) hand side of the equation. This means that both side is equal to the same constant[7] $-\bar{\beta}^2$:

$$\frac{1}{\widetilde{A}}\left[2i\beta_0\left(\partial_z\widetilde{A}\right) - \beta_0^2\widetilde{A}\right] = -\bar{\beta}^2 \tag{5.33a}$$

$$-\frac{1}{F(x,y)}\left[\triangle_\perp F(x,y) + \epsilon(\omega)k_0^2 F(x,y)\right] = -\bar{\beta}^2 \tag{5.33b}$$

yielding

$$\triangle_\perp F(x,y) + \left[\epsilon(\omega)k_0^2 - \bar{\beta}\right] F(x,y) = 0 \tag{5.34a}$$

$$2i\beta_0\partial_z\widetilde{A} + \left(\bar{\beta}^2 - \beta_0^2\right) = \widetilde{A} \tag{5.34b}$$

In this set of equations the wavenumber $\bar{\beta}$ is determined by solving the eigenvalue equation for the waveguide (see App. A). We should also point that

$$\epsilon(\omega) = \left[n(\omega) + \Delta n\right]^2 \simeq n^2 + 2n\Delta n \tag{5.35}$$

where $\Delta n = \bar{n}_2 |E|^2 + ({}^{i\alpha}/_{2k_0})$ is a small quantity that takes into account the nonlinear effect and the losses (α). The remaining task is to solve the set of eq. (5.34).

Strategy of the calculation: Since the influence of the nonlinear effect is minute we can use perturbation theory to solve the eq. (5.34a):

1. $\bar{\beta}$ is the wavenumber of the waveguide without any nonlinear effect. It is found by solving the eigenvalue equation of the given waveguide. Of course this wavenumber depends on the frequency of the propagating field $\bar{\beta} = \beta(\omega)$
2. As a result from the nonlinearity the wavenumber is slightly modified such that $\bar{\beta}(\omega) = \beta(\omega) + \delta\beta$. It can be shown that

$$\delta\beta = \frac{\omega^2 n(\omega)}{c^2\beta(\omega)} \frac{\iint \Delta n \left|F(x,y)\right|^2 dxdy}{\iint \left|F(x,y)\right|^2 dxdy} \tag{5.36}$$

3. Regarding the evolution of the envelope (eq. (5.34b)) we can notice that

$$\bar{\beta}^2 - \beta_0^2 = \left(\bar{\beta} - \beta_0\right)\left(\bar{\beta} + \beta_0\right) \simeq 2\beta_0\left(\bar{\beta} - \beta_0\right) \tag{5.37}$$

and therefore the envelope equation is now

$$\partial_z\widetilde{A} = i\left[\bar{\beta}(\omega) - \beta_0\right]\widetilde{A} = i\left[\beta(\omega) + \delta\beta - \beta_0\right]\widetilde{A} \tag{5.38}$$

$\Rightarrow$ Each frequency component within the pulse envelope acquires a phase-shift, whose magnitude depends on both the frequency - because of $\beta(\omega)$ - and the intensity - because $\delta\beta$ depends on $|E|^2$. We can now proceed as previously by Taylor-expanding the wavenumber

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + o\left[(\omega - \omega_0)^3\right] \tag{5.39}$$

[7] β is the wavenumber and the reason to choose the sign of the constant is actually dictated by the final result!

and use the relation[8]

$$(\partial_t)^\alpha \;\Leftrightarrow\; [-i(\omega-\omega_0)]^\alpha \tag{5.40}$$

yielding

$$\partial\widetilde{A} = -\beta_1\partial_t\widetilde{A} - \frac{i\beta_2}{2}\partial_{tt}\widetilde{A} + i\delta\beta\widetilde{A} \tag{5.41}$$

As we did for the nonlinear Schrodinger equation (eq. (5.26), we can introduce the reduced time T moving with the pulse at its group velocity v_G

$$T = t - \frac{z}{v_G} = t - \beta_1 z$$

so that finally we obtain

$$\boxed{\frac{\partial\widetilde{A}}{\partial z} = -\frac{\alpha}{2}\widetilde{A} - i\frac{\beta_2}{2}\frac{\partial^2\widetilde{A}}{\partial T^2} + i\gamma\left|\widetilde{A}\right|^2\widetilde{A}} \tag{5.42}$$

where the nonlinear coefficient γ is[9]

$$\gamma = \frac{n_2\omega}{cA_{\text{eff}}} \qquad \text{with } A_{\text{eff}} = \frac{\left(\iint |F(x,y)|^2\,dxdy\right)^2}{\iint |F(x,y)|^4\,dxdy} \tag{5.43}$$

5.3 Pulse propagating in a fibre

The goal of this section if to analyse the different contribution (linear and nonlinear) on the pulse that are described by the nonlinear Schrodinger equation (5.42)

5.3.1 Linear effect: the dispersion

In a purely dispersive material the Schrodinger equation simply becomes

$$i\frac{\partial U}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 U}{\partial T^2} \tag{5.44}$$

which can be solved by using the so-called *Fourier-transform method*, which is based on the definition of the pulse from its spectrum

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widetilde{U}(z,\omega)e^{-i\omega T}d\omega \tag{5.45}$$

and allows transforming the eq. (5.44) in the frequency domain by using

$$\mathcal{F}\left[\partial_z U(z,T)\right] = \partial_z\widetilde{U}(z,T) \tag{5.46a}$$

$$\mathcal{F}\left[(\partial_T)^n U(z,T)\right] = (-i\omega)^n\,\widetilde{U}(z,T) \tag{5.46b}$$

[8]This comes from the choice of the convention for the Fourier transformation:

$$A(z,t) = \frac{1}{2\pi}\int\widetilde{A}(z,\omega-\omega_0)e^{-i(\omega-\omega_0)t}d\omega$$

[9]Note that in this equation the nonlinear refractive index n_2 is averaged over the active area:

$$\bar{n}_2 \to \frac{n_2}{\iint |F(x,y)|^2\,dxdy}$$

The equation (5.44) is then

$$i\frac{\partial \widetilde{U}(z,\omega)}{\partial z} = -\frac{1}{2}\beta_2\omega^2\widetilde{U}(z,\omega) \tag{5.47}$$

which is readily solved as

$$\widetilde{U}(z,\omega) = \widetilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\ \omega^2 z\right) \tag{5.48}$$

where $\widetilde{U}(0,\omega)$ is the spectrum of the pulse at $z = 0$. It is clear from the eq. (5.48) that the spectrum in the initial pulse is ***not modified*** during the propagation. As this equation suggests each spectral component ω acquires a different phase during the propagation. The temporal shape of the pulse after a distance z is then

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\ \omega^2 z - i\omega T\right)d\omega \tag{5.49}$$

where $\widetilde{U}(0,\omega)$ is the inverse Fourier transform of the pulse at $z = 0$

$$\widetilde{U}(0,\omega) = \int_{-\infty}^{+\infty} U(0,T)e^{i\omega T}dT \tag{5.50}$$

5.3.2 Propagation of a Gaussian pulse

As an example we consider here that we have a Gaussian pulse propagating in a dispersive medium. Its envelope is given by

$$U(0,T) = \exp\left(-\frac{T^2}{2\ T_0^2}\right) \tag{5.51}$$

where T_0 is the half-width (at $^1\!/_e$ -intensity point). Although this definition makes the calculations easier but it is in practice more convenient to work with the full-width at half-maximum (FWHM) instead of T_0

$$T_{\text{FWHM}} = 2\sqrt{\ln 2}\ T_0 \approx 1.665\ T_0 \tag{5.52}$$

By inserting the eq. (5.51) into eq. (5.49) and calculating the integration we obtain the shape of the pulse at a distance z:

$$U(z,T) = \frac{T_0}{\sqrt{T_0^2 - i\beta_2 z}}\exp\left[-\frac{T^2}{2\,(T_0^2 - i\beta_2 z)}\right] \tag{5.53}$$

which is still a Gaussian pulse. However the amplitude and the duration of the pulse are modified. In particular the exponential can be written as

$$\begin{aligned}\exp\left[\frac{T^2}{2\,(T_0^2 - i\beta_2 z)}\right] &= \exp\left[-\frac{T^2\,(t_0^2 + i\beta_2 z)}{2\,(T_0^4 + \beta_2^2 z^2)}\right] \\ &= \exp\left[-\frac{T_0^2 T^2}{2\,(T_0^4 + \beta_2^2 z^2)}\right]\exp\left[-i\frac{\beta_2 z T^2}{2\,(T_0^4 + \beta_2^2 z^2)}\right]\end{aligned} \tag{5.54}$$

From the first exponential we find that the pulse is actually getting broader. Its duration is now given by

$$T^2(z) = \frac{T_0^4 + \beta_2^2 z^2}{T_0^2} \quad \rightarrow \quad \boxed{T(z) = T_0\sqrt{1+\left(\frac{z}{L_D}\right)}} \tag{5.55}$$

where the *dispersion length* $L_D = T_0^2/|\beta_2|$. This length is a sort of scale for the minimal length necessary to observe the effect of dispersion. For a propagation distance $L > L_D$ the effect of dispersion can be observed. We could notice that this scaling factor was somehow already visible from the eq. (5.44) by analysing the dimensions:

$$\begin{aligned} i\frac{\partial U}{\partial z} &= -\frac{1}{2}\beta_2\frac{\partial^2 U}{\partial T^2} \\ \Rightarrow \quad \frac{[U]}{[\text{length}]} &= [\beta_2]\frac{[U]}{[\text{time}^2]} \;\Rightarrow\; \text{typical length} \propto \frac{[\text{time}^2]}{\beta_2} \end{aligned} \tag{5.56}$$

It is clear that the shorter the pulse the shorter the dispersion length. Therefore, for a given physical length of the dispersion element, short pulses will be more affected by dispersion than long pulses. In the context of laser cavity, this means that the effect of dispersion will be more severe when the targeted pulses are short.

Obviously the appearance of complex number in the expression of $U(z,T)$ in both the amplitude and the exponential part (eq. (5.53)) implies that the phase is modified as the pulse propagates. We can rewrite $U(z,T)$ as

$$U(z,T) = |U(z,T)| \exp\left[i\phi(z,T)\right] \tag{5.57}$$

where the phase $\phi(z,T)$ is then given by

$$\phi(z,T) = \frac{-\text{sgn}(\beta_2)\,(z/L_D)}{1+(z/L_D)^2}\frac{T^2}{2T_0^2} + \frac{1}{2}\tan^{-1}\left(\frac{z}{L_D}\right) \tag{5.58}$$

Since the phase depends on time the instantaneous frequency will differ at different location in the pulse from its central frequency ω_0. This difference of instantaneous frequency ($\delta\omega$) is called *the chirp* and is defined (using eq. (5.58)) as[10]

$$\delta\omega = -\frac{\partial\phi}{\partial T} = \frac{\text{sgn}(\beta_2)\,(z/L_D)}{1+(z/L_D)^2}\frac{T}{2T_0^2} \tag{5.59}$$

It is clear that the instantaneous frequency changes linearly across the pulse. This is a *linear chirp*. For normal dispersion regime ($\beta_2 > 0$) $\delta\omega$ increases from negative ($T < 0$) value to positive ones. Obviously in anomalous dispersion regime the situation is reversed.

5.3.3 The nonlinear phase-shift due to the Kerr effect

In this part we are only interested by the effect of the nonlinear term in the NLSE (eq. (5.42)) which now simply reads as

$$i\frac{\partial \widetilde{A}}{\partial z} = -\gamma\left|\widetilde{A}\right|^2\widetilde{A} \tag{5.60}$$

[10]The minus sign in the definition of the instantaneous frequency - and therefore the chirp - comes from the choice of definition for the electric field eq. (5.1).

Introducing the peak intensity P_0 we can replace $\widetilde{A}$ with $\sqrt{P_0}U$, where U is dimensionless so as to write the equation as

$$i\frac{\partial U}{\partial z} = -\gamma P_0 \left|U\right|^2 U \tag{5.61}$$

As previously we can do a simple analysis of the dimension to see that there must be a scaling length related to the nonlinear effect. This length is the *nonlinear length*

$$L_{\mathrm{NL}} = \frac{1}{\gamma P_0} \tag{5.62}$$

To solve the eq. (5.61) we introduce the Ansazt $U = V \exp i\phi_{\mathrm{NL}}$ into the equation:

$$i\frac{\partial V}{\partial z}e^{i\phi_{\mathrm{NL}}} - \frac{\partial \phi_{\mathrm{NL}}}{\partial z}Ve^{i\phi_{\mathrm{NL}}} = -\frac{V^2}{L_{\mathrm{NL}}}Ve^{i\phi_{\mathrm{NL}}} \tag{5.63}$$

and we can separate real and imaginary parts:

$$\frac{\partial \phi_{\mathrm{NL}}}{\partial z} = \frac{V^2}{L_{\mathrm{NL}}} \tag{5.64a}$$

$$\frac{\partial V}{\partial z} = 0 \tag{5.64b}$$

From the imaginary part we see that the amplitude will not change during the propagation. The phase on the other hand will vary as a function of the propagation distance:

$$\phi_{\mathrm{NL}} = \frac{V^2}{L_{\mathrm{NL}}}z \tag{5.65}$$

The solution of the propagation equation (5.61) is therefore

$$\boxed{U(z,T) = U(0,T)\exp\left[i\phi_{\mathrm{NL}}(z,T)\right]} \text{ with } \phi_{\mathrm{NL}} = \left|U(0,T)\right|^2 \frac{z}{L_{\mathrm{NL}}} \tag{5.66}$$

The central part of the pulse ($T = 0$) is where the maximum phase shift occurs. Is is given by

$$\phi_{\max} = \frac{L_{\mathrm{eff}}}{L_{\mathrm{NL}}} = \gamma P_0 L_{\mathrm{eff}} \tag{5.67}$$

where L_{eff} is the effective lenght of the fibre, which takes the loss into account. Fig. 5.3 shows the effect of self-phase modulation on an initial unchirped Gaussian pulse as it propagates and acquires a nonlinear phase shift.

As it propagates the pulse acquires an intensity-dependent phase-shift but its shape (in the time domain) remains unchanged! As previously we can define the chirp linked with the Kerr effect:

$$\delta\omega = -\frac{\partial \phi}{\partial T} = -\left(\frac{z}{L_{\mathrm{NL}}}\right)\frac{\partial}{\partial T}\left|U(0,T)\right|^2 \tag{5.68}$$

By contrast with the dispersion effects the frequency chirp depends on the shape of the initial pulse, and also its initial chirp. The magnitude of the chirp increases during propagation[11]. This also means that as the pulse propagates its spectrum continuously broadens. In the case of an initial Gaussian pulse the resulting chirp is

$$U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right) \Rightarrow \delta\omega(T) = \left(\frac{T}{T_0}\right)\frac{z}{L_{\mathrm{NL}}}\exp\left(-\frac{T^2}{2T_0^2}\right) \tag{5.69}$$

[11] In the case of dispersion the factor z cancels out.

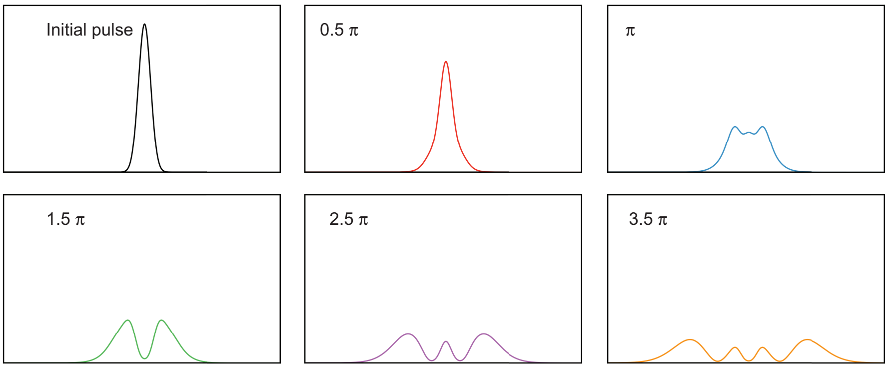

Figure 5.3: Spectral broadening occuring on an initialy unchurped Gaussian pulse due to self-phase modulation. The propagation lenght is such that the maximum phase-shift ϕ_{rmmax} is indicated for each subplot. Adapted from ref. [1].

Note that in the vicinity of the centre of the pulse ($T \sim 0$) the SPM-induced chirp is almost linear and positive. As we saw previously the dispersion yields either a linear positive or negative chirp depending of the regime of dispersion that we considered. In the anomalous dispersion regime it is possible to find a regime where the SPM-induced chirp and the linear contribution can perfectly balance each other. In that case the pulse is called a *soliton*.

5.4 Interplay between dispersive and nonlinear effects

5.4.1 Modulational instability

Modulational instability is a very common type of instability that can naturally occur as a result of an interplay between the dispersive effects and the nonlinearity of the system. We can find such instability in many fields such as hydrodynamics, plasma physics or optics. As all instabilities, this results in an exponential growth of an input perturbation. To study this, the idea is to apply a small perturbation $a(z, T)$ on the steady state solution $\bar{A} = \sqrt{P_0} \exp(i\phi_{\mathrm{NL}})$ of the nonlinear Schrodinger equation. In eq. (5.26) we insert the perturbed amplitude

$$A \rightarrow \left(\sqrt{P_0} + a\right) e^{i\phi_{\mathrm{NL}}} \text{ with } \phi_{\mathrm{NL}} = \gamma P_0 z \tag{5.70}$$

The different terms for the NLSE are[12]

$$\partial_z A = \left[\partial_z a + i\gamma P_0 \left(\sqrt{P_0} + a\right)\right] e^{i\phi_{\text{NL}}} \tag{5.71a}$$

$$\partial_{TT} A = \partial_{TT} a e^{i\phi_{\text{NL}}} \tag{5.71b}$$

$$|A|^2 A = \left[P_0\sqrt{P_0} + aP_0 + P_0(a+a^*)\right] e^{i\phi_{\text{NL}}} \tag{5.71c}$$

and finally we obtain

$$i\left[\partial_z a + i\gamma P_0 \left(\sqrt{P_0} + a\right)\right] = \frac{\beta_2}{2}\left(\partial_{TT} a\right) - \gamma P_0 \left[\sqrt{P_0} + a + (a+a^*)\right]$$

$$\implies \quad i\partial_z a = \frac{\beta_2}{2}\partial_{TT} a - \gamma P_0 (a + a^*) \tag{5.72}$$

Note that the presence of $(a + a^*)$ suggest a solution of the type

$$a = a_1 e^{i\theta} + a_2 e^{-i\theta} \qquad \text{with} \theta = Kz - \Omega T \tag{5.73}$$

where K and Ω are respectively the wavenumber and the frequency of the perturbation. We should insert this expression in eq. (5.72) yielding

$$-Ka_1 e^{i\theta} + Ka_2 e^{-i\theta} = \frac{\beta_2}{2}\left(-\Omega^2 a_1 e^{i\theta} - \Omega^2 a_2 e^{-i\theta}\right) - \gamma P_0 \left(a_1 e^{i\theta} + a_2 e^{-i\theta} + a_1 e^{-i\theta} + a_2 e^{i\theta}\right) \tag{5.74}$$

which can simply written in the form

$$\begin{aligned} \left(\frac{\beta_2}{2}\Omega^2 + \gamma P_0 - K\right) a_1 + \gamma P_0 a_2 &= 0 \\ \gamma P_0 a_1 + \left(\frac{\beta_2}{2}\Omega^2 + \gamma P_0 + K\right) a_2 &= 0 \end{aligned}$$

and this has a non-trivial solution if $\det(\dots) = 0$:

$$\begin{aligned} \left(\frac{\beta_2}{2}\Omega^2 + \gamma P_0 - K\right)\left(\frac{\beta_2}{2}\Omega^2 + \gamma P_0 + K\right) - \gamma^2 P_0^2 &= 0 \\ \left(\frac{\beta_2}{2}\Omega^2 + \gamma P_0\right)^2 - K^2 - \gamma^2 P_0^2 &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad K^2 &= \frac{1}{4}\beta_2^2\Omega^4 + \cancel{\gamma^2 P_0^2} + \beta_2\Omega^2\gamma P_0 - \cancel{\gamma^2 P_0^2} \\ &= \frac{1}{4}\beta_2\Omega^2\left(\Omega^2 + 4\frac{\gamma P_0}{\beta_2}\right) \end{aligned} \tag{5.76}$$

[12] Remember that the contributions of order $o\left(a^2\right)$ are neglected:

$$\begin{aligned} |A|^2 A &= \left(\sqrt{P_0} + a\right)\left(\sqrt{P_0} + a^*\right)\left(\sqrt{P_0} + a\right) e^{i\phi_{\text{NL}}} \\ &= \left[P_0 + (a+a^*)\sqrt{P_0} + \cancel{|a|^2}\right]\left(\sqrt{P_0} + a\right) e^{i\phi_{\text{NL}}} \\ &= \left[P_0\sqrt{P_0} + P_0 a + P_0(a+a^*) + \cancel{a(a+a^*)\sqrt{P_0}}\right] e^{i\phi_{\text{NL}}} \end{aligned}$$

Finally we have the wavenumber of the perturbation

$$K = \frac{1}{2}\,|\beta_2\Omega|\,\sqrt{\Omega^2 + \mathrm{sign}(\beta_2)\Omega_c^2} \tag{5.77}$$

where we introduced $\Omega_c^2 = \left({}^{4\gamma P_0}/_{|\beta_2|}\right) = \left({}^{4}/_{|\beta_2|\,L_{\mathrm{NL}}}\right)$. Of course we should remember that we had factor out $\exp\left[-i(\omega_0 T - \beta_0 z\right]$ in eq. (5.30) and therefore the frequency and the wavenumber of the perturbation are $\beta_0 \pm K$ and $\omega_0 \pm \Omega$ respectively. As we can see from eq. (5.77), operating in normal ($\beta_2 > 0$) or in anomalous ($\beta_2 < 0$) can yield different behaviours for the perturbation. Indeed in anomalous dispersion regime, K can become imaginary, yielding an exponential growth of the perturbation[13]. Setting $\mathrm{sign}(\beta_2) = -1$ we have the gain $g(\Omega) = 2\,\mathrm{Im}(K)$:

$$g(\Omega) = |\beta_2\Omega|\,\sqrt{\Omega_c^2 - \Omega^2} \tag{5.78}$$

Note that gain only exists for $\Omega_c^2 - \Omega > 0$ and the maximal value of that gain is obtained for

$$\Omega_{\mathrm{max}} = \pm\frac{\Omega_c}{\sqrt{2}} = \pm\sqrt{\frac{2\gamma P_0}{|\beta_2|}} \tag{5.79}$$

and its value is

$$g_{\mathrm{max}} = g\left(\Omega_{\mathrm{max}}\right) = \frac{1}{2}\,|\beta_2|\,\Omega_c^2 = 2\gamma P_0 \tag{5.80}$$

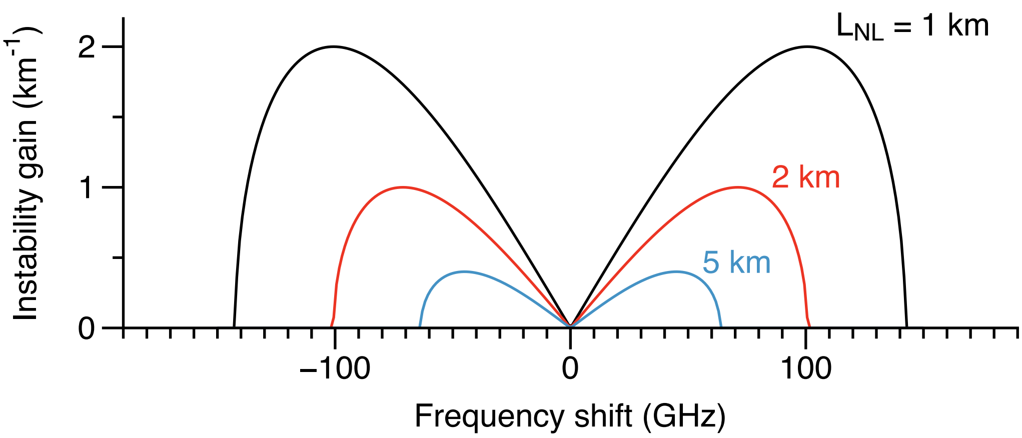

Figure 5.4: Gain spectra for modulational instability for three value of nonlinear length L_{NL}. The dispersion is $\beta_2 = -5\mathrm{ps}^2/\mathrm{km}$. Reproduced from ref.[1].

[13] There is already an i in the exponential where K appear.

Appendix A

Optical modes in fibre

A.1 The eigenvalue problem

In a perfectly transparent dielectric material (no loss) we can write from Maxwell equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{A.1a}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{A.1b}$$

$$\tag{A.1c}$$

where $\epsilon = \epsilon_0 n^2$ with n the refractive index of the material. Moreover since we are considering here optical fibre we should use the operator $\nabla = \left(\frac{\partial}{\partial r}; \ \frac{1}{r}\frac{\partial}{\partial \theta}; \ \frac{\partial}{\partial z} \right)$ yielding the different components of the electric field

$$\frac{1}{r}\frac{\partial e_z}{\partial \theta} + i\beta e_\theta = -i\omega\mu_0 h_r \tag{A.2a}$$

$$-i\beta e_r - \frac{\partial e_z}{\partial r} = -i\omega\mu_0 h_\theta \tag{A.2b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r e_\theta) - \frac{1}{r}\frac{\partial e_r}{\partial \theta} = -i\omega\mu_0 h_z \tag{A.2c}$$

and the magnetic field

$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} + i\beta h_\theta = +i\omega\epsilon_0 n^2 e_r \tag{A.3a}$$

$$-i\beta h_r - \frac{\partial h_z}{\partial r} = +i\omega\epsilon_0 n^2 e_\theta \tag{A.3b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r h_\theta) - \frac{1}{r}\frac{\partial h_r}{\partial \theta} = +i\omega\epsilon_0 n^2 e_z \tag{A.3c}$$

From eq. (A.2b) and (A.3a) we can extract h_θ

$$-i\beta e_r - \frac{\partial e_z}{\partial r} = -i\omega\mu_0 h_\theta \qquad \times\, i\beta \tag{A.4a}$$

$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} + i\beta h_\theta = i\omega\epsilon_0 n^2 e_r \qquad \times\, (-i\omega\mu_0) \tag{A.4b}$$

$$\Rightarrow \quad -i\left(\beta\frac{\partial e_z}{\partial r}+\frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial\theta}\right)=\left(\omega^2\epsilon_0 n^2\mu_0-\beta^2\right)e_r \tag{A.5}$$

Similarly we can express the different transverse components of the electric and the magnetic field as a function of the longitudinal part of the field e_z and h_z:

$$e_r=\frac{-i}{k^2n^2-\beta^2}\left(\beta\frac{\partial e_z}{\partial r}+\frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial\theta}\right) \tag{A.6a}$$

$$e_\theta=\frac{-i}{k^2n^2-\beta^2}\left(\frac{\beta}{r}\frac{\partial e_z}{\partial\theta}-\omega\mu_0\frac{\partial h_z}{\partial r}\right) \tag{A.6b}$$

$$h_r=\frac{-i}{k^2n^2-\beta^2}\left(\beta\frac{\partial h_z}{\partial r}-\frac{\omega\epsilon_0 n^2}{r}\frac{\partial e_z}{\partial\theta}\right) \tag{A.6c}$$

$$h_\theta=\frac{-i}{k^2n^2-\beta^2}\left(\frac{\beta}{r}\frac{\partial h_z}{\partial\theta}+\omega\epsilon_0 n^2\frac{\partial e_z}{\partial r}\right) \tag{A.6d}$$

What is remarkable from these equations is that the knowledge of e_z and h_z is sufficient to calculate the whole profile. Since we have cylindrical symmetry it is a good practise to express the electromagnetic fields as

$$\mathbf{E}=\mathbf{e}(r,\theta)e^{i\omega t-\beta z} \tag{A.7a}$$

$$\mathbf{H}=\mathbf{h}(r,\theta)e^{i\omega t-\beta z} \tag{A.7b}$$

Moreover these fields must fulfil the propagation equation that we express according to the Helmholtz equation

$$\nabla^2\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right\}+\left(\frac{n\omega}{c}\right)^2\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right\}=0 \tag{A.8}$$

In this equation we should use the Laplacian in cylindrical coordinate[1] so as to obtain the propagation equations for each three components of the fields. In principle we should deal with all six components $\{\mathbf{E},\mathbf{H}\}_{(r,\theta,z)}$. However $\mathbf{E}$ and $\mathbf{H}$ must also fulfil the Maxwell's equation yielding only two independent components. Traditionally we choose the z-components (eq. (A.6)). The equations to consider are therefore:

$$\partial_{rr}e_z+\frac{1}{r}\partial_r e_z+\frac{1}{r^2}\partial_{\theta\theta}e_z+\left[k^2n^2(r,\theta)-\beta^2\right]e_z=0 \tag{A.9a}$$

$$\partial_{rr}h_z+\frac{1}{r}\partial_r h_z+\frac{1}{r^2}\partial_{\theta\theta}h_z+\left[k^2n^2(r,\theta)-\beta^2\right]h_z=0 \tag{A.9b}$$

the remaining problem is to solve these equation to obtain e_z, h_z and use the eq. (A.6) to obtain the whole electromagnetic field. Note that in that equation the refractive index can simply be written as $n(r,\theta)=n(r)$ because of the symmetry of the system.

[1]We remind that for a function $f(r,\theta,z)$ the Laplacian is given by

$$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}$$

In the present case, considering the expression for the field (eq. (A.7a)), the second derivative ∂_{zz} yields a $-\beta^2$ term.

A.1.1 Fibre modes

There exist three types of modes depending on the value of e_z and h_z:

1. if $e_z = 0$ we talk about TE mode.
2. if $h_z = 0$ we talk about TM mode.
3. if neither e_z nor h_z is null then the mode is called an *hybrid mode*.

TE modes

They correspond to the case when $e_z = 0$. This obviously strongly simplify the set of eq.(A.6)

$$e_r = \frac{-i\omega\mu_0}{k^2n^2 - \beta^2}\frac{1}{r}\frac{\partial h_z}{\partial \theta} \tag{A.10a}$$

$$e_\theta = \frac{i\omega\mu_0}{k^2n^2 - \beta^2}\frac{\partial h_z}{\partial r} \tag{A.10b}$$

$$h_r = \frac{-i\beta}{k^2n^2 - \beta^2}\frac{\partial h_z}{\partial r} \tag{A.10c}$$

$$h_\theta = \frac{-i\beta}{k^2n^2 - \beta^2}\frac{1}{r}\frac{\partial h_z}{\partial \theta} \tag{A.10d}$$

but also the propagation equations (A.9). Indeed since $e_z = 0$ we only need to deal with the propagation equation applied to the magnetic field $\mathbf{H}$ (eq. (A.9b)). Moreover in order to take the cylindrical symmetry of the problem the magnetic field must be expressed as a function of $\cos(p\theta + \phi)$ or $\sin(p\theta + \phi)$ where $p \in \mathbb{Z}$ and ϕ is a constant phase term.

$$h_z = \left\{ \begin{array}{c} g(r) \\ h(r) \end{array} \right\} \cos(p\theta + \phi) \text{ or } \; h_z = \left\{ \begin{array}{c} g(r) \\ h(r) \end{array} \right\} \sin(p\theta + \phi) \tag{A.11}$$

where $g(r)$ is used to describe the field in the core region and $h(r)$ in the cladding region. Note that in both cases such dependence yield a p^2 in the propagation equation:

$$\frac{\partial^2 h_z}{\partial r^2} + \frac{1}{r}\frac{\partial h_z}{\partial r} + \left(k^2n(r)^2 - \beta^2 - \frac{p^2}{r^2}\right) h_z = 0 \tag{A.12}$$

The boundary condition requires that $g(a) = h(a)$ and using the cosine dependence on the magnetic field in eq. (A.11) in the eq. (A.12) yields

$$\frac{i\beta}{k^2n_{cl}^2 - \beta^2}\frac{p}{a}h(a)\sin(p\theta + \phi) = \frac{i\beta}{k^2n_{co}^2 - \beta^2}\frac{p}{a}g(a)\sin(p\theta + \phi) \tag{A.13}$$

which only holds if $p = 0$, which means that there is actually no dependence in θ! Since e_r and h_θ only depends on $\partial_\theta h_z$ there are both null: $e_r = 0$ and $h_\theta = 0$.
As we see the original is considerably simplified:

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \left(k^2n(r)^2 - \beta^2\right) h_z = 0 \tag{A.14a}$$

$$e_\theta = \frac{i\omega\mu_0}{k^2n^2 - \beta^2}\frac{dh_z}{dr} \tag{A.14b}$$

$$h_r = \frac{-i\beta}{k^2n^2 - \beta^2}\frac{dh_z}{dr} \tag{A.14c}$$

$$e_r = h_r = 0 \tag{A.14d}$$

In the core we have to solve

$$\frac{d^2h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \alpha^2 h_z = 0 \quad \text{with } \alpha^2 = k^2 n_{co}^2 - \beta^2 \tag{A.15}$$

for with the solution[2] is the Bessel function of 0^{th}-order $J_0(\alpha r)$.
In the core we have to solve

$$\frac{d^2h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} - \gamma^2 h_z = 0 \quad \text{with } \gamma^2 = \beta^2 - k^2 n_{cl}^2 \tag{A.16}$$

for with the solution[3] is the modified Bessel function of 1^{st}-order $I_0(\gamma r)$.

In conclusion we have the longitudinal component of the magnetic field inside the core and the cladding region:

$$h_z = \begin{cases} AJ_0(\alpha r) & \forall\, |r| \le a \\ BK_0(\gamma r) & \forall r > a \end{cases} \tag{A.17}$$

where A and B are two constant to be determined. The continuity of the field components h_z and e_θ at the boundary $r = a$ lead to

$$AJ_0(\alpha a) = BK_0(\gamma a) \tag{A.18a}$$

$$\frac{A}{\alpha} J_0'(\alpha a) = -\frac{B}{\gamma} K_0'(\gamma a) \tag{A.18b}$$

$$\Rightarrow \frac{J_0'(\alpha a)}{\alpha J_0(\alpha a)} = -\frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} \quad \Leftrightarrow \quad \frac{J_0'(U)}{UJ_0(U)} = -\frac{K_0'(W)}{WK_0(W)} \tag{A.19}$$

where we use the parameters $U = \alpha a$ and $W = \gamma a$ that are similar to the ones we introduced for the planar waveguide. Solving this eigenvalue problem yields the propagation constant β. Note that we can also use the relationship

$$J_0'(x) = -J_1(x) \tag{A.20a}$$

$$K_0'(x) = -K_1(x) \tag{A.20b}$$

Then we can rewrite the eigenvalue problem as

$$\boxed{\frac{J_1(U)}{UJ_0(U)} = -\frac{K_1(W)}{WK_0(W)}} \tag{A.21}$$

Strategy: The strategy[4] to obtain the modes in the fibre is the following:

1. solve the eigenvalue problem for a given set of parameter $\{a, n_{co}, n_{cl}\}$
2. Use the value of β to express the longitudinal components of the fields e_z and h_z
3. Use the longitudinal components to calculate the transverse components of the field. In the particular case of TE mode there are e_θ and h_r.

[2]Mathematically the 0^{th}-order Neumann function $N_0(\alpha r)$ is also a solution of this differential equation but it diverges at $r = 0$ and therefore cannot physical be suitable.

[3]Mathematically the modified Bessel function of 2^{nd} kind $K_0(\alpha r)$ is also a solution of this differential equation but it diverges at $r \to \infty$ and therefore cannot physical be suitable.

[4]Obviously this strategy is general and does not only apply to the TE mode.

In the particular case of TE mode the field in the core and cladding are

$$\text{in the core:} \begin{cases} e_\theta &= -i\omega\mu_0 \frac{a}{U} A J_1\left(\frac{U}{a}r\right) \\ h_r &= i\beta \frac{a}{U} A J_1\left(\frac{U}{a}r\right) \\ h_z &= A J_0\left(\frac{U}{a}r\right) \end{cases} \tag{A.22a}$$

$$\text{in the cladding:} \begin{cases} e_\theta &= i\omega\mu_0 \frac{a}{W} A K_1\left(\frac{W}{a}r\right) \\ h_r &= -i\beta \frac{a}{W} \frac{J_0(U)}{K_0(W)} A K_1\left(\frac{U}{a}r\right) \\ h_z &= \frac{J_0(U)}{K_0(W)} A K_0\left(\frac{W}{a}r\right) \end{cases} \tag{A.22b}$$

Note that the constant B in eq. (A.17) is replaced by $\left({}^{J_0(U)}/_{K_0(W)}\, A\right)$ calculated from boundary conditions at the interface core – cladding. To evaluate the constant A we use the **z**-component of the Poynting vector

$$S_z = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{\mathbf{z}} = \frac{1}{2}\left(e_r H_\theta^* - e_\theta h_r^*\right) \tag{A.23}$$

and evaluate the optical power

$$P = \iint S_z \cdot r dr d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\left(e_r H_\theta^* - e_\theta h_r^*\right)\cdot r dr d\theta \tag{A.24}$$

TM modes This is the very same idea expect that this time $h_z = 0$. This yields that $e_\theta = h_r = h_z = 0$. In that case the eigenvalue equation is

$$\boxed{\frac{J_1(U)}{U J_0(U)} = -\left(\frac{n_{cl}}{n_{co}}\right)^2 \frac{K_1(W)}{W K_0(W)}} \tag{A.25}$$

and various components of the electromagnetic field

$$\text{in the core:} \begin{cases} e_r &= i\beta \dfrac{a}{U} A J_1\left(\dfrac{U}{a} r\right) \\ e_z &= A J_0\left(\dfrac{U}{a} r\right) \\ h_\theta &= i\omega\epsilon_0 n_{co}^2 A\ J_1\left(\dfrac{U}{a} r\right) \end{cases} \tag{A.26a}$$

$$\text{in the cladding:} \begin{cases} e_r &= -i\beta \dfrac{a}{W} \dfrac{J_0(U)}{K_0(W)} A K_1\left(\dfrac{W}{a} r\right) \\ e_z &= \dfrac{J_0(U)}{K_0(W)} A K_0\left(\dfrac{W}{a} r\right) \\ h_\theta &= -i\omega\epsilon_0 n_{cl}^2 \dfrac{a}{W} \dfrac{J_0(U)}{K_0(W)}\ K_1\left(\dfrac{W}{a} r\right) \end{cases} \tag{A.26b}$$

Hybrid modes In the case when neither e_z nor h_z is null[5] the solutions of the propagation equation eq. (A.9a) and (A.9b) will be given as a product of the n^{th}-order Bessel function and the azimuthal dependence $\cos(n\theta+\phi)$ or $\sin(n\theta+\phi)$ where $n \in \mathbb{Z}$. Since now both propagation equations are coupled we cannot simply eliminate the angular dependence ($n \neq 0$). Note however that the azimuthal dependence has to remain when crossing the interface core–cladding. Moreover in the expression of e_r, e_θ, h_r and h_θ appear terms as $\partial_r e_z$, $\partial_\theta e_z$ ... the azimuthal dependencies for the electric and the magnetic fields must follow

$$e_z = \begin{cases} A J_n\left(\dfrac{U}{a} r\right) \cos(n\theta+\phi) & \text{in the core} \\ A \dfrac{J_n(U)}{K_n(W)} K_n\left(\dfrac{W}{a} r\right) \cos(n\theta+\phi) & \text{in the core} \end{cases} \tag{A.27a}$$

$$h_z = \begin{cases} C J_n\left(\dfrac{U}{a} r\right) \sin(n\theta+\phi) & \text{in the core} \\ C \dfrac{J_n(U)}{K_n(W)} K_n\left(\dfrac{W}{a} r\right) \sin(n\theta+\phi) & \text{in the core} \end{cases} \tag{A.27b}$$

which we can use to express the e_r, e_θ and h_r, h_θ:

[5] The continuity conditions still need to hold!

- in the core ($|r| \leq a$):

$$e_r = \frac{-ia^2}{U^2}\left[A\beta\frac{U}{a}J_n'\left(\frac{U}{a}r\right) + C\omega\mu_0\frac{n}{r}J_n\left(\frac{U}{a}r\right)\right]\cos\left(n\theta+\phi\right) \tag{A.28a}$$

$$e_\theta = \frac{-ia^2}{U^2}\left[-A\beta\frac{n}{r}J_n\left(\frac{U}{a}r\right) - C\omega\mu_0\frac{u}{a}J_n'\left(\frac{U}{a}r\right)\right]\sin\left(n\theta+\phi\right) \tag{A.28b}$$

$$h_r = \frac{-ia^2}{U^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{n}{r}J_n\left(\frac{U}{a}r\right) + C\beta\frac{U}{a}J_n'\left(\frac{U}{a}r\right)\right]\sin\left(n\theta+\phi\right) \tag{A.28c}$$

$$h_\theta = \frac{-ia^2}{U^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{n}{r}J_n'\left(\frac{U}{a}r\right) + C\beta\frac{U}{a}J_n\left(\frac{U}{a}r\right)\right]\cos\left(n\theta+\phi\right) \tag{A.28d}$$

- in the cladding ($r > a$):

$$e_r = \frac{ia^2}{W^2}\left[A\beta\frac{W}{a}K_n'\left(\frac{W}{a}r\right) + C\omega\mu_0\frac{n}{r}K_n\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\cos\left(n\theta+\phi\right) \tag{A.29a}$$

$$e_\theta = \frac{ia^2}{W^2}\left[-A\beta\frac{n}{r}K_n\left(\frac{W}{a}r\right) - C\omega\mu_0\frac{W}{a}K_n'\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\sin\left(n\theta+\phi\right) \tag{A.29b}$$

$$h_r = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{co}^2\frac{n}{r}K_n\left(\frac{W}{a}r\right) + C\beta\frac{W}{a}K_n'\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\sin\left(n\theta+\phi\right) \tag{A.29c}$$

$$h_\theta = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{co}^2\frac{n}{r}K_n'\left(\frac{W}{a}r\right) + C\beta\frac{n}{r}K_n\left(\frac{W}{a}r\right)\right]\frac{J_n(U)}{K_n(W)}\cos\left(n\theta+\phi\right) \tag{A.29d}$$

Additionally we need to have continuity of the e_θ and h_θ at $r = a$, which leads to

$$A\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n = -C\omega\mu_0\left[\frac{J_n'(U)}{UJ_n(U)} + \frac{K_n'(W)}{WK_n(U)}\right] \tag{A.30a}$$

$$A\omega\epsilon_0\left[n_{cl}^2\frac{J_n'(U)}{UJ_n(U)} + n_{co}^2\frac{K_n'(W)}{WK_n(U)}\right] = -C\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n \tag{A.30b}$$

which has a solution for $\{A, C\}$ if and only if

$$\boxed{\begin{aligned}\left[\frac{J_n'(U)}{UJ_n(U)} + \frac{K_n'(W)}{WK_n(U)}\right]&\left[n_{cl}^2\frac{J_n'(U)}{UJ_n(U)} + n_{co}^2\frac{K_n'(W)}{WK_n(U)}\right] \\ &= \frac{\beta^2}{k^2}\left(\frac{1}{U^2}+\frac{1}{W^2}\right)n^2 = \left(\frac{n_{co}^2}{U^2}+\frac{n_{cl}^2}{W^2}\right)n^2\end{aligned}} \tag{A.31}$$

which is the eigenvalue equation that we need to solve in order to find the propagation constant β. In this equation n is a positive integer number. As previously the equation must be solve numerically for a given value of the normalised frequency V.

On Fig. A.1, we plot the eigenvalue equation for a 4μm-core diameter rod, which can be seen as an optical fibre with a silica core and an air-cladding. In that case the *normalised parameter* is $V = 12.395$. As we can see, this curve has several zeros, each of which corresponds to one particular value of U and therefore to one propagation constant β. As a general form the modes are labelled as HE_{nm} and EH_{nm} where n is the integer number used in eq. (A.31), corresponding to the order of the Bessel function used, and m is the index of the considered zero. Note though the the modes are alternating: the first one is HE_{11}. Then comes an EH mode and so on.

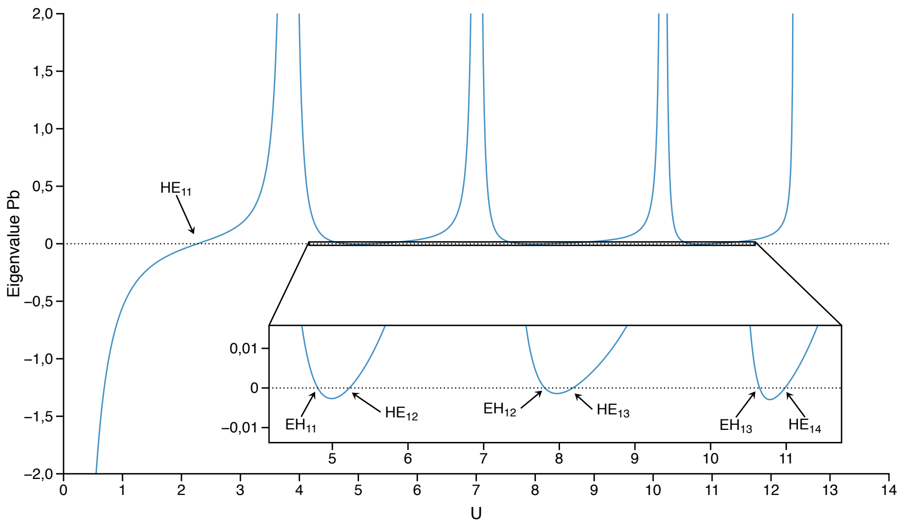

Figure A.1: Eigenvalue problem eq. (A.31) for a fibre with a diameter $2a = 4\mu$m. The wavelength is 1064 nm. This corresponds to $V = 12.395$ for the considered waveguide (silica core surrounding by air). The arrow indicates the zeros and the corresponding optical modes.

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