Modern Optics
Scalar wave optics

0.1 Maxwell's equations

Maxwell's equation are the basis for a proper description of an electromagnetic wave. They link the different quantities:

$\mathbf{E}$:	electric field	$[E] = V \cdot m^{-1}$
$\mathbf{H}$:	magnetic field	$[H] = A \cdot m^{-1}$
$\mathbf{D}$:	electric displacement	$[D] = A \cdot s \cdot m^{-2}$
$\mathbf{B}$:	magnetic induction	$[B] = V \cdot s \cdot m^{-2}$
$\mathbf{j}$:	electric current density	$[J] = A \cdot m^{-2}$
ρ :	electric charge density	$[\rho] = A \cdot s \cdot m^{-3}$

Note that all quantities are function of the spatial coordinate (x, y, z) as well as the time t. Using the nabla operator

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Maxwell equations are formulated as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

The meaning of each equation is quite clear:

(1a) the Faraday law: The variation of the magnetic induction $\mathbf{B}$ creates vortexes of the electric field.

(1b) the Ampere law: voices of the magnetic field $\mathbf{H}$ are either caused by an electric current with a density $\mathbf{j}$ or by a temporal variation of the electric displacement $\mathbf{D}$. $\nabla_t \mathbf{D}$ is the electric displacement current.

(1c) the Gauss law: the sources of the electric displacement $\mathbf{D}$ are the electric charges with density ρ.

(1d) the magnetic field (induction) is solenoidal, *i.e., there exists no* magnetic charges.

To these equations are associated the so-called *constitutive relations* for the material:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} \tag{2a}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu_0 \mu_r \mathbf{H} \tag{2b}$$

where ϵ_r (resp. μ_r) is called the relative *permitivity* (resp. *permeability*). Note that in the most general case ϵ_r is a tensor. For non magnetic materials $\mu_r = 1$ and can be dropped. The vacuum permitivity ϵ_0 and vacuum permeability μ_0 are given by

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1} \simeq \frac{1}{36\pi \times 10^{-6}} \qquad \left(\mathrm{A} \cdot \mathrm{s}/\mathrm{V} \cdot \mathrm{m}\right) \tag{3}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H} \cdot \mathrm{m}^{-1} \simeq 1.256 \times 10^{-6} \qquad \left(\mathrm{V} \cdot \mathrm{s}/\mathrm{A} \cdot \mathrm{m}\right) \tag{4}$$

0.1.1 Propagation in vacuum

In vacuum ($\epsilon_r = 1$) and for non-magnetic materials the equations become

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{5a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{5b}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{5c}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{5d}$$

Using the algebraic equation $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.A) - \nabla^2 \mathbf{A}$ applied to eq. (5a), we derive the equation for the propagation of the electric field[1]:

$$\left[\nabla^2 - \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = \underbrace{\nabla\left(\rho/\epsilon_0\right) + \mu_0 \frac{\partial \mathbf{J}}{\partial t}}_{\text{source terms}} \tag{6}$$

Finally, in an isotropic material without any charge nor loss (no density of current) the equation of propagation becomes:

$$\nabla^2 E - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0 \tag{7}$$

Note that in many situations we can derive a similar equation for the propagation of the magnetic field $\mathbf{B}$. It is therefore legitimate, at this stage, to generalize the equation of propagation by

$$\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0 \tag{8}$$

where $U(\mathbf{r}, t)$ is the *wave function* and can either represent the electric field $\mathbf{E}$ or the magnetic field $\mathbf{B}$ and the constant c is such that $\epsilon_0 \mu_0 c^2 = 1$. Note that we could write the eq. (8) as

$$\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) U = 0, \tag{9}$$

[1] To find the equation of propagation of the field from Maxwell's equation we need to use the algebraic identity:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

applied to eq. (5a):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla\left(\rho/\epsilon_0\right) - \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \nabla\left(\rho/\epsilon_0\right) - \mu_0 \frac{\partial}{\partial t}\left(\nabla \times \mathbf{H}\right)$$

and use the eq. (5b).

which has two solutions as contra-propagating waves $U = U_F + U_B$ with:

$$U_F = \psi\left(t - \frac{z}{c}\right) \tag{10a}$$

$$U_B = \psi\left(t + \frac{z}{c}\right) \tag{10b}$$

where U_F is a forward propagating wave and U_B the backward-propagating wave. Note that the function ψ does not need to be harmonic. From this formulation it is obvious that c is the velocity of the wave. It is actually the phase velocity. The speed of the wave (phase-velocity) is given by

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \tag{11}$$

0.1.2 Linear propagation in a material

For an isotropic material, we can simply replace ϵ_0 by $(1 + \chi_e)\epsilon_0 = \epsilon_r \epsilon_0$ and obtain the equation of propagation

$$\nabla^2 U - \epsilon_r \epsilon_0 \frac{\partial^2 U}{\partial t^2} = \nabla^2 U - \frac{\epsilon_r}{c^2} \frac{\partial^2 U}{\partial t^2} = 0 \tag{12}$$

The phase velocity is in that case:

$$v_\varphi = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{1 + \chi_e}} = \frac{c}{n} \tag{13}$$

where n is the refractive index of the material.

0.2 Flow of energy

Poynting vector

An electromagnetic wave can carry energy and it is important to evaluate this by using the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. From the dimension aspect we have

$$[\mathbf{S}] = \mathrm{V} \cdot \mathrm{m}^{-1} \times \mathrm{A} \cdot \mathrm{m}^{-1} = \frac{\mathrm{A} \cdot \mathrm{V}}{\mathrm{m}^2} = \frac{\mathrm{W}}{\mathrm{m}^2} \tag{14}$$

The Poynting vector has the dimension of a power over and area: this represents an intensity! The norm of the Poynting vector $|\mathbf{S}|$ is the flow of energy per unit area and unit time: this really represents the transport of the energy carried by the EM field. To evaluate the total power available we should integrate the Poynting vector over a close area (Fig. 0.2). The calculation can be done with the help of the Green-Ostrogradsky theorem:

$$\oint\!\!\oint \mathbf{S} \cdot \mathbf{n}\, dA = \iiint_V \nabla \cdot \mathbf{S}\, dV \tag{15}$$

where we can evaluate the divergence of the Poynting vector

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{16}$$

by using the Maxwell-Ampère (eq. (1b)) and Maxwell-Faraday (eq. (1a)) equations:

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{17}$$

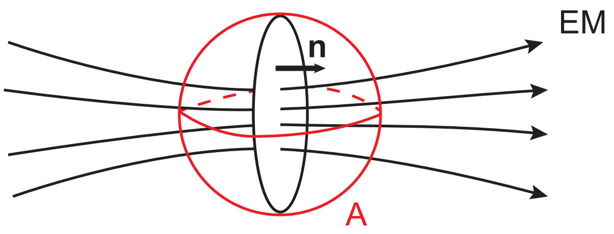

Figure 1: Use of the Green-Ostrogradsky theorem to evaluate the flow of the Poynting vector through a closed area A.

And for dielectrics ($\rho = 0$, $\mathbf{J} = 0$) we obtain

$$\begin{aligned}\nabla \cdot \mathbf{S} &= -\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= -\left(\epsilon_0 \epsilon_r \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mu_R \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}\right) \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 \epsilon_r \mathbf{E} \cdot \mathbf{E} + \mu_0 \mu_r \mathbf{H} \cdot \mathbf{H}\right) \end{aligned} \tag{18}$$

Using the eq. (18) and eq. (15) leads to

$$\oiint \mathbf{S} \cdot \mathbf{n}\, dA = P_{\text{net}} = -\frac{\partial}{\partial t} \iiint \underbrace{\frac{1}{2}\,\epsilon_0 \epsilon_r\ \mathbf{E} \cdot \mathbf{E}}_{\substack{W_E \\ \text{elect. energy dens.}}} + \underbrace{\frac{1}{2}\,\mu_0 \mu_r\ \mathbf{H} \cdot \mathbf{H}}_{\substack{W_H \\ \text{magnet. energy dens.}}} \, dV \tag{19}$$

where P_{net} is the net power and the *minus*-sign indicates that the amount of the energy in the volume decreases over time it the net amount of power flowing through the surface is positive: the energy flows out of the surface.

Intensity and amplitude

Since the electric field and the magnetic fields are time-dependant so is the Poynting vector. Using the general form

$$U = \frac{1}{2} U_0 e^{-i\omega t} + c.c. = \text{Re}\left[U_0 e^{-i\omega t}\right] \tag{20}$$

the Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\left(\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{i\omega t}\right) \times \frac{1}{2}\left(\mathbf{H}_0 e^{-i\omega t} + \mathbf{H}_0^* e^{i\omega t}\right) \\ &= \frac{1}{4}\left(\mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0 + e^{-2i\omega t} \mathbf{E}_0 \times \mathbf{H}_0 + e^{2i\omega t} \mathbf{E}_0^* \times \mathbf{H}_0^*\right)\end{aligned} \tag{21}$$

In fact a detector will not resolve the fast oscillation of the Poynting vector by time-average it

$$\langle \mathbf{S} \rangle_T = \frac{1}{4}\left(\mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0\right) = \frac{1}{2}\left(\mathbf{S} + \mathbf{S}^*\right) \tag{22}$$

with $\mathbf{S} = {}^1\!/_2\ \mathbf{E} \times \mathbf{H}^*$ is the complex Poynting vector. The intensity of the beam is then $|\mathbf{S}| = {}^1\!/_2\ E_0 H_0 = {}^1\!/_2\ E_0 \left({}^{E_0}\!/_{\eta}\right)$ where $\eta = \sqrt{{}^{\mu}\!/_{\epsilon}}$ is the impedance of the medium. In the case of a non-magnetic material $\mu = \mu_0$ and $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 n^2$ and therefore

$$I = \frac{1}{2} \frac{|E_0|^2}{\eta} = \frac{1}{2} n \sqrt{\frac{\epsilon_0}{\mu_0}}\, |E_0|^2 = \frac{1}{2} n \sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{\epsilon_0 \mu_0 c^2}\, |E_0|^2$$

$$\Rightarrow \quad \boxed{I = \frac{1}{2} n c \epsilon_0 \left| E_0 \right|^2} \tag{23}$$

This gives the relationship between the amplitude of the electric field E_0 and the measured intensity I.

0.3 Solutions of the propagation equation

0.3.1 Helmholtz equation

As we saw previously, the propagation of an electromagnetic wave in vacuum is described by the equation

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \tag{24}$$

Note that in the current case the phase velocity is c and it is constant. There is no dispersion here. Since this is a spatio-temporal equation (partial differential equation), it is legitimate to try solving it by separating the variable:

$$\mathbf{U}(\mathbf{r}, t) = A(\mathbf{r}) f(t) \tag{25}$$

where $A(\mathbf{r})$ only depends on the spatial coordinate $\mathbf{r}$ and $f(t)$ is a function of the time. Inserting this ansatz into eq. (24) yields

$$\left[\nabla^2 A f(t) - \frac{1}{c^2} A(\mathbf{r}) \frac{\partial^2 f(t)}{\partial t^2}\right] \cdot A(\mathbf{r}) f(t) = 0$$

$$\Rightarrow \qquad \frac{\boldsymbol{\nabla}^2 A(\mathbf{r})}{A(\mathbf{r})^2} = \frac{1}{c^2}\frac{1}{f(t)}\frac{\partial^2 f(t)}{\partial t^2} \tag{26}$$

Since each side of the eq. (26) only depends on one variable, they must be both equal to the same constant value. For convenience we take this value equal to $-k^2$. Let us first look at the time-dependent function $f(t)$. It must be solution of the differential equation

$$\frac{1}{c^2}\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} = -k^2 \quad \Rightarrow \quad \frac{d^2 f}{dt^2} + k^2 c^2 f = 0 \tag{27}$$

This has a periodic solution of the form $\exp(\pm i\omega t)$ where $\omega^2 = k^2 c^2$ is the angular frequency of the function $f(t)$. Here we will choose the $-$ sign as a convention. $k = \left({}^{2\pi}/_{\lambda}\right)$ is the *wave-number*. The LHS of the eq. (26) on the other hand yields the so-called *Helmholtz equation*:

$$\boxed{\left[\nabla^2 + k^2\right] A(\mathbf{r}) = 0} \tag{28}$$

The relation between k and ω is called the dispersion relation. It is interesting to note that the Helmholtz equation takes care of the dispersion and is valid even with the phase velocity is not constant.

0.3.2 The plane wave solution

The easiest solution of the Helmholtz equation is the plane wave

$$A(\mathbf{r}) = A_0 e^{i\mathbf{k}\cdot\mathbf{r}} \tag{29}$$

where A_0 is the complex envelope and is a constant value and $\mathbf{k} = (k_x, k_y, k_z)$ is the wave-vector indicating the direction of propagation of the wave. The phase-front of the that wave is simply given by $\varphi(\mathbf{r}$ =constant, corresponding to $\mathbf{k}\cdot\mathbf{r}$ =constant. These are planes. The norm of $\mathbf{k}$ is the wave-number $|\mathbf{k}|$ such that $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 = (\omega/c)^2$.

Since we took the convention that $f(t) \propto \exp(-i\omega t)$ the wave described by eq. (29) is traveling in the forward direction. Along $z-$direction it is :

$$E = \frac{1}{2}(U + U^*) = |A_0| \cos[kz - \omega t + \arg(A_0)] \tag{30}$$

Of course such wave is an idealized solution since the intensity $I(r) \propto |A_0|^2$ is constant everywhere! Note that we can write the argument of the cosine function as $(z/c - t)\,\omega$, which clearly shows that we have a wave traveling in the $+z$ direction (forward direction) at a velocity c. The Helmholtz equation has another plane wave solution $\propto \exp[-i(\omega t + kz)]$ corresponding to a wave traveling along z but in the backward direction.

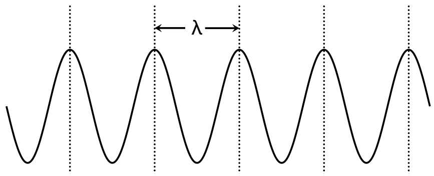

Figure 2: representation of a plane wave

In a material with a refractive index n, the phase-velocity is $v_\varphi = c/n$ and the wavelength is scaled by n such that $\lambda_{\text{mat.}} = \lambda_0/n$. In the optics domain $\lambda \in [400, 800]$ nm (visible). This yields the optical frequency $\nu \in [375, 750]$ THz or the angular frequency $\omega = 2\pi\nu \in [2.3 \times 10^{15}, 4.7 \times 10^{15}]$ rad/s.

0.3.3 The spherical wave

If we solve the Helmholtz equation in spherical coordinate[2] we can find solution as a spherical wave[3]

$$A(\mathbf{r}) = A_0 \frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|} \tag{31}$$

which is a wave traveling from a point source. It originates from a source and its wavefront consist of spheres centered about the point source (Fig. 3). As previously due to the convention of time ($f \propto \exp(-i\omega t)$) the wave described with the complex amplitude presented on eq. (31) travels from the source point outwardly. On the other hand a spherical wave traveling inwards (towards the point source) is described by

$$A(\mathbf{r}) = A_0 \frac{e^{-ik|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|} \tag{32}$$

[2]We remind that the Laplacian operator in spherical coordinate is given by

$$\nabla^2 A = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial A}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 A}{\partial\varphi^2}$$

[3]It is important to note the difference in the argument of the exponential. In the case of a spherical wave we have a simple product kr and in the case of the plane wave this a scalar product $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$.

The intensity of the spherical wave decays with the distance from the point source as $I(r) = \left({|A_0|}/{r^2}\right)$.

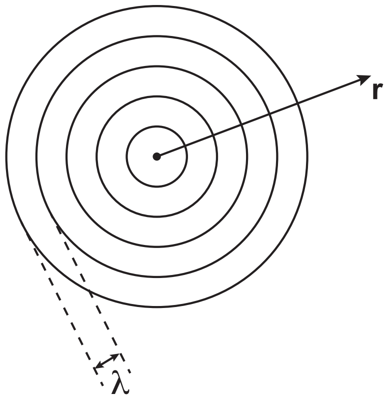

Figure 3: Representation of a spherical wave.

0.3.4 The paraxial wave and parabolic equation

Laser beams (Gaussian beams) can neither be described by plane waves not by spherical waves. A collimated laser beam propagates straight although it diverges due to diffraction during the propagation. It then behaves along the direction of propagation ($z-$axis) nearly as a plane wave instead of having a constant amplitude like the plane wave its amplitude depends on (x, y) and is spatially localized around the $z-$axis.
Mathematically such a wave is described as[4]

$$\mathbf{E}(x, y, z, t) = \frac{1}{2}\, A(\mathbf{r})e^{-i\omega t}\mathbf{n} + c.c. \tag{33}$$

where

$$A(\mathbf{r}) = F(x, y, z)e^{ikz} \tag{34}$$

In this form $F(x, y, z)$ is the complex amplitude. Note that in this form we can assume that the phase term evolve at a different rate as the complex amplitude. In particular the paraxial wave must fulfill the following condition: after a distance $\Delta z = \lambda$ the change $\Delta F(x, y, z)$ is much smaller than $F(x, y, z)$ itself. Since the evolution of $F(x, y, z)$ along the propagation is

$$\Delta F = \left(\frac{\partial F}{\partial z}\right)\Delta z = \left(\frac{\partial F}{\partial z}\right)\lambda \tag{35}$$

then $\Delta A \ll A \Leftrightarrow \left({\partial A}/{\partial z}\right) \ll \left({A}/{\lambda}\right) = \left({Ak}/{2\pi}\right)$. Such approximation is called the *paraxial approximation*. Obviously within this approximation, the first derivative should also vary slowly within the distance λ so that in conclusion we can write

$$\left|\frac{\partial F}{\partial z}\right| \ll k\,|F| \tag{36a}$$

$$\left|\frac{\partial^2 F}{\partial z^2}\right| \ll k^2\,|F| \tag{36b}$$

[4]we assume here that the beam is linearly polarised field and we simply use the scalar form

Inserting eq. (34) into the eq. (28) yields[5]

$$(\partial_{xx} + \partial_{yy})F + \partial_{zz}F + 2ik\partial_z F = 0 \tag{38}$$

And using the paraxial approximation, *i.e.* neglecting $\partial_{zz}F$ in comparison with $k\partial_z F$ we finally obtain the *parabolic* equation, also called the *paraxial Helmholtz equation*:

$$\triangle_\perp F + 2ik\partial_z F = 0 \tag{39}$$

This equation is the fundamental equation to study the Gaussian optics. Note that we can write this equation in the following form:

$$\boxed{\frac{\partial F}{\partial z} = \frac{i}{2k}\triangle_\perp F} \tag{40}$$

In this form, we clearly see that as the beam propagates it experiences the action of the transverse Laplacian $\triangle_\perp$, which is taking into account the diffraction of the beam. An interesting solution of this equation is the Gaussian beam. We leave this for a lecture on Laser.

0.3.5 Gaussian beam

One solution of the parabolic equation (eq. (40) is the Gaussian beam. Because of its symmetry the ∇- operator has to be expressed in cylindrical coordinate which yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = -2ik\frac{\partial F}{\partial z} \tag{41}$$

It can be shown that one solution of eq. (41) is a Gaussian beam. The fundamental mode (TEM_{00}) ix expressed by

$$\psi(r) = e^{-i\left[p(z) + \frac{kr^2}{2q(z)}\right]} \tag{42}$$

where $p(z)$ is the complex phase which varies with the propagation and $q(z)$ the complex curvature ($q \in \mathbb{C}$). The complex parameter q can be expressed as a function of the radius of curvature of the wavefront R and the size of the beam $w(z)$:

$$\frac{1}{q} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w(z)^2} \tag{43}$$

and $P(z)$ is the so-called Gouy phase.

5

$$\left(\nabla^2 + k^2\right)A = 0 \;\Leftrightarrow\; \left[\partial_{xx} + \partial_{yy} + \partial_{zz}\right]Fe^{ikz} + k^2Fe^{ikz} = 0$$
$$\text{and } \partial_{zz}(Fe^{ikz}) = (\partial_{zz}F)e^{ikz} + 2ik\partial_z Fe^{ikz} - k^2Fe^{ikz}$$

0.4 Propagation of a wave between two materials

0.4.1 Fresnel formulae

As we know already from Snell-Descartes' law of refraction, as the light changes from one medium to another the direction of the **k**-vector, in other word, the direction of propagation of the wave is modified according to

$$n_1 \sin\theta_i = n_2 \sin\theta_2 \tag{44}$$

for the transmitted beam and $\theta_r = \pi - \theta_i$ for the reflected beam.

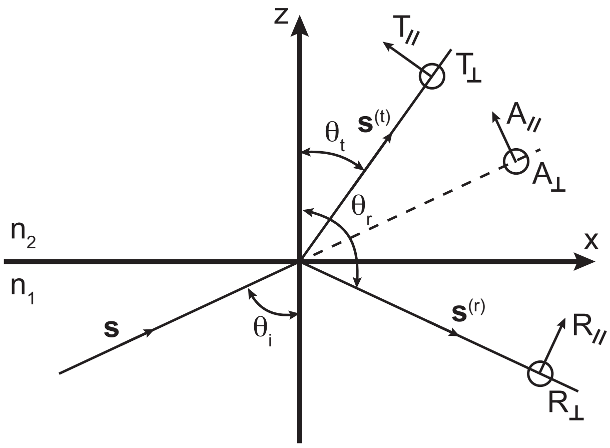

Figure 4: Transmission and reflection of an incoming beam at an interface between two material with refractive indices n_1 and n_2. Subscript (r) indicates the reflected beam and (t) the transmitted one. In that case $n_1 > n_2$.

If we assume that both media are isotropic and homogeneous and perfectly transparent and non magnetic. Let A be the complex amplitude of the electric field so that this field is described by

$$\mathbf{E} = \begin{pmatrix} E_x^{(i)} & = & -A_{\|} \cos\theta_i \; e^{-i\phi_i} \\ E_y^{(i)} & = & A_{\perp} \; e^{-i\phi_i} \\ E_z^{(i)} & = & A_{\|} \sin\theta_i \; e^{-i\phi_i} \end{pmatrix} \tag{45}$$

where the variable phase is given by

$$\phi_i = \omega \left(t - \frac{\mathbf{s}\cdot\mathbf{r}}{v_i} \right) = \omega \left(t - \frac{x \sin\theta_i + z\cos\theta_i}{v_i} \right) \tag{46}$$

where v_i is the phase-velocity of the incoming beam, therefore in the material n_1. The boundary conditions applied to the electric field and the magnetic field, which is easily deduced from eq. (45) at the interface yield the set of equations:

$$\left(A_{\|} - R_{\|}\right)\cos\theta_i = T_{\|}\cos\theta_t \tag{47a}$$
$$A_{\perp} + R_{\perp} = T_{\perp} \tag{47b}$$
$$\sqrt{\epsilon_1}\left(A_{\perp} - R_{\perp}\right)\cos\theta_i = \sqrt{\epsilon_2}\; T_{\perp}\cos\theta_t \tag{47c}$$
$$\left(A_{\|} + R_{\|}\right)\sqrt{\epsilon_1} = T_{\|}\;\sqrt{\epsilon_2} \tag{47d}$$

Those clearly indicate that there exist two types of waves. These equations are indeed either related to the transverse components only (eq. (47b) and (47c)) or to the parallel components (eq. (47a) and (47d)). These waves correspond to the two possible polarization of the incoming wave. Considering that $n = \sqrt{\epsilon}$, we can extract the transmission and reflection coefficients:

$$T_{\parallel} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} A_{\parallel} \tag{48a}$$

$$T_{\perp} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} A_{\perp} \tag{48b}$$

$$R_{\parallel} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} A_{\parallel} \tag{48c}$$

$$R_{\perp} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} A_{\parallel} \tag{48d}$$

Of course for normal incidence these lead to the well-known formulae

$$T_{\parallel} = \frac{2}{n+1} A_{\parallel} \ ; \quad T_{\perp} = \frac{2}{n+1} A_{\perp} \tag{49a}$$

$$R_{\parallel} = \frac{n-1}{n+1} A_{\parallel} \ ; \quad R_{\perp} = -\frac{n-1}{n+1} A_{\perp} \tag{49b}$$

where $n = {}^{n_2}/_{n_1}$.

0.4.2 Total internal reflection

What happens when the light propagates from a material into one with a lower refractive index? Starting from the situation presented on Fig. 4, suppose that the beam is now coming from the medium 2 and we still have $n_2 > n_1$.

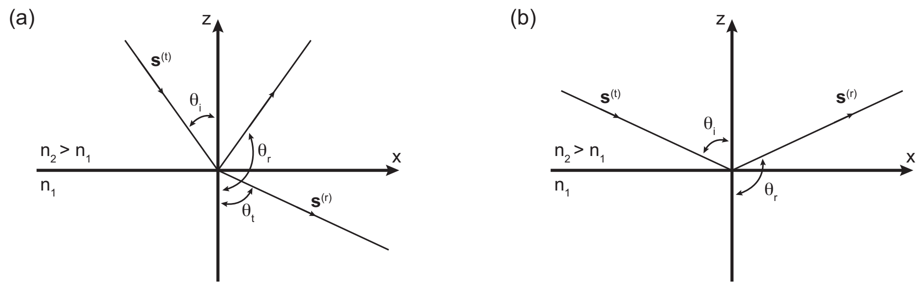

Figure 5: (a) refraction into a material with a lower refractive index ($n_1 < n_2$) and (b) total internal reflection.

According to the Snell's law (eq. (44)) the larger the incident angle θ_i, the larger should be the refracted angle... until θ_t eventually reaches ${}^{\pi}/_{2}$! When θ_i is equal to the so-called *critical angle* θ_c such that $\theta_t = {}^{\pi}/_{2}$ the wave is totally reflected (Fig. 5b). This critical angle is defined by

$$\sin\theta_c = \frac{n_2}{n_1} = n_{12} = n \tag{50}$$

Note that above this angle there is no light entering the second medium. Actually the EM wave has not disappear from the second material, but there is no longer any flow of energy across the boundary. From the Snell's law we have

$$\sin\theta_t = \frac{\sin\theta_i}{n} \Rightarrow \cos\theta_t = \pm i\sqrt{\frac{\sin^2\theta_i}{n^2} - 1} \tag{51}$$

and since the phase of the transmitted beam (eq. (46) and Fig. 4) is

$$\phi_t = \omega\left(t - \frac{x\sin\theta_t + z\cos\theta_t}{v_2}\right) = \omega\left(t - \frac{x\sin\theta_t}{v_2} - \frac{z\cos\theta_t}{v_2}\right) \tag{52}$$

Using eq. (51) we see that the phase term for the transmitted wave is

$$e^{-i\phi_t} = \exp\left[-i\omega\left(t - \frac{x\sin\theta_i}{nv_2}\right)\right]\exp\left[\mp\frac{\omega z}{v_2}\sqrt{\frac{\sin^2\theta_i}{n^2} - 1}\right] \tag{53}$$

This is an inhomogeneous beam, which travels along the $x-$direction - therefore along the boundary - but which varies exponentially with the distance z from the boundary. Physically we should keep the *minus* sign in the eq. (53). By inserting the eq. (51) in the Fresnel's coefficient we obtain

$$R_{\parallel} = \frac{n_2\cos\theta_i - in_1\sqrt{\frac{\sin^2\theta_i}{n^2} - 1}}{n_2\cos\theta_i + in_1\sqrt{\frac{\sin^2\theta_i}{n^2} - 1}} A_{\parallel}$$

and a similar equation for $R_{\perp}$. After rearranging a bit we obtain

$$R_{\parallel} = \frac{n_2^2\cos\theta_i - in_1^2\sqrt{\sin^2\theta_i - n^2}}{n_2^2\cos\theta_i + in_1^2\sqrt{\sin^2\theta_i - n^2}} A_{\parallel} \tag{54a}$$

$$R_{\perp} = \frac{\cos\theta_i - in_1^2\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + in_1^2\sqrt{\sin^2\theta_i - n^2}} A_{\perp} \tag{54b}$$

from which we can extract the phase-shift ϕ_j associated with the reflection for each polarization j by expressing[6] the ratio $R_j/A_j = \exp(i\phi_j)$:

$$\phi_{\parallel} = -2\ \arctan\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right) \tag{55a}$$

$$\phi_{\perp} = -2\ \arctan\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right) \tag{55b}$$

[6] It is noticeable that for both polarization we have $R_j/A_j = z/z^*$ where $z \in \mathbb{C}$. As a consequence we can write $\exp(i\phi_j)$ as $\exp(2i\alpha_j)$, where $\alpha_j = \arg(z)$. This directly leads to

$$\tan\frac{\phi_{\parallel}}{2} = -\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i} \quad \text{and} \quad \tan\frac{\phi_{\perp}}{2} = -\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}$$

.1 Integral formulation of the Maxwell equations

The form that we gave for the Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{56a}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{56b}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{56c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{56d}$$

is true at any point of the space. Since we are dealing with vector fields it can be easier to use the integral formulation to appreciate better the meaning of those equations. For this we remind the vectorial relations:

$$\oiint_S \mathbf{A} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{A} \, dV \tag{57a}$$

$$\oint_C \mathbf{A} \, d\ell = \iint_S \nabla \times \mathbf{A} \, dS \tag{57b}$$

The eq. (57a) is the Green-Ostrogradsky theorem (also called the *divergence* theorem) and the eq. (57b) is the Stokes' theorem. Let apply the Stokes' theorem to the eq. (56c) and eq. (56d):

$$\iiint \nabla \cdot \mathbf{E} \, dV = \iiint \frac{\rho}{\epsilon_0} dV \quad \Rightarrow \quad \oiint_S \mathbf{E} \cdot \mathbf{n} \, dS = \frac{Q}{\epsilon_0} \tag{58a}$$

$$\iiint \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \oiint_S \mathbf{B} \cdot \mathbf{n} \, dS = 0 \tag{58b}$$

where Q is the electric charge enclosed in the volume V. The eq. (58a) is the integral formulation of the *Gauss theorem*. It tells that the flux of the electric field out through a closed surface is equal to the total number of charge contained inside the closed surface S. The eq. (58b) on the other hand implies that an equivalent charge for the magnetic field cannot exist: there is no magnetic monopole.

Let now apply the Green-Ostrogradsky theorem on eq. (56a) and eq. (56a). We obtain

$$\iint \nabla \times \mathbf{E} \, dS = -\iint \frac{\partial B}{\partial t} \, dS \quad \Rightarrow \quad \oint_C \mathbf{E} \, d\ell = -\frac{\partial \Phi(B)}{\partial t} \tag{59a}$$

$$\iint \nabla \times \mathbf{H} \, dS = \iint \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \, dS \quad \Rightarrow \quad \oint_C \mathbf{B} \, d\ell = \mu_0 I_S + \mu_0 \frac{\Phi_S(E)}{\partial t} \tag{59b}$$

where $\Phi_S(E)$ (resp. $\Phi_S(B)$) is the flux of the electric (resp. magnetic) field through the surface S. As we see from eq. (59a) the variation of the flux of the magnetic field induces a current in a circuit surrounding that surface. This is the *Faraday's law*. The eq. (59a) on the other hand shows that a constant (I_S) flowing through a surface will induce a magnetic field circulating around that flow. This is the *Ampère law*. Actually the second term is a correction added by Maxwell: the variation of flux of the electric field through a surface will yield a magnetic field just like the variation of the flux of magnetic field induced a current.

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