$$\tilde{E}(\vec{r}, \omega-\omega_0) = \underbrace{F(x,y)}_{\text{transverse distribution of the mode}} \underbrace{\tilde{A}(z, \omega-\omega_0)}_{\text{amplitude}} e^{i\beta_0 z}$$

this must be solution of the Helmholtz equation: $\nabla^2 \tilde{E} + \varepsilon(\omega) k_0 \tilde{E} = 0$

let's calculate $\nabla^2 \tilde{E} = (\partial_{zz} + \Delta_\perp^2)\tilde{E}$

$$\partial_{zz}\left[F(x,y)\tilde{A}(z,\omega-\omega_0)e^{i\beta_0 z}\right] = F(x,y)\,\partial_z\left[(\partial_z A)e^{i\beta_0 z} + i\beta_0 \tilde{A} e^{i\beta_0 z}\right]$$

$$= F(x,y)\left\{(\partial_{zz}A)e^{i\beta_0 z} + 2i\beta_0(\partial_z \tilde{A})e^{i\beta_0 z} - \beta_0^2 \tilde{A} e^{i\beta_0 z}\right\}$$

Therefore the Helmholtz eq. becomes:

$$F(x,y)\left[\underbrace{\cancel{(\partial_{zz}\tilde{A})e^{i\beta_0 z}}}_{\text{SVEA}} + 2i\beta_0(\partial_z\tilde{A})e^{i\beta_0 z} - \beta_0^2\tilde{A}e^{i\beta_0 z}\right] + \Delta_\perp^2 F(x,y)\,\tilde{A}(z,\omega-\omega_0)\,e^{i\beta_0 z} + \varepsilon(\omega)k_0\tilde{A}e^{i\beta_0 z} = 0$$

$$\Leftrightarrow \quad F(x,y)\left(2i\beta_0(\partial_z\tilde{A}) - \beta_0^2\tilde{A}\right) = -\tilde{A}\left[(\partial_{xx}+\partial_{yy})F(x,y) + \varepsilon(\omega)k_0 F(x,y)\right]$$

$$\Leftrightarrow \quad \frac{1}{\tilde{A}}\left(2i\beta_0(\partial_z\tilde{A}) - \beta_0^2\tilde{A}\right) = -\frac{1}{F(x,y)}\left[(\partial_{xx}+\partial_{yy})F(x,y) + \varepsilon(\omega)k_0 F(x,y)\right]$$

Since $\tilde{A}$ & $F(x,y)$ are independent and each side of this eq. only depends on either $F(x,y)$ or $\tilde{A}$ then both side must be equal to the same constante:

$$\begin{cases} -\dfrac{1}{\tilde{A}}\left[2i\beta_0(\partial_z A) - \beta_0^2\tilde{A}\right] = +\bar{\beta}^2 \\[2ex] \dfrac{1}{F(x,y)}\left[(\partial_{xx}+\partial_{yy})F(x,y) + \varepsilon(\omega)k_0 \cancel{\beta_0} F(x,y)\right] = \bar{\beta}^2 \end{cases}$$

$$\boxed{\begin{array}{c}(\partial_{xx}+\partial_{yy})\,F(x,y) + (\varepsilon(\omega)\,k_0 - \bar{\beta}^2)\,F(x,y) = 0 \\ 2i\beta_0\,\partial_z \tilde{A} + (\bar{\beta}^2 - \beta_0^2)\,\tilde{A} = 0\end{array}}$$

the wavenumber $\bar{\beta}$ is determined by solving the eigenvalue eq. for the fiber modes.

Note that $\varepsilon(\omega) = (n+\Delta n)^2 \simeq n^2 + 2n\Delta n$ since Δn is small $\qquad \Delta n = n_2 |E|^2 + \frac{i\alpha}{2k_0}$

$\Rightarrow$ to solve $\Delta_\perp F + (\varepsilon(\omega)\,k_0 - \bar{\beta}^2)\,F = 0$ we use perturbation theory:

<u>step 1</u>: we find $\bar{\beta}$ without non linear effect $\rightarrow \bar{\beta} = \beta(\omega)$

<u>step 2</u>: Δn induces a change in $\bar{\beta}(\omega)$ as $\bar{\beta}(\omega) = \beta(\omega) + \delta\beta$

and it can be shown that $\quad \delta\beta = k_0 \dfrac{\iint \Delta n\,|F(x,y)|^2\,dx\,dy}{\iint |F(x,y)|^2\,dx\,dy}$

Regarding the eq. for the envelope: we use $\bar{\beta}^2 - \beta_0^2 \simeq (\bar{\beta}-\beta_0)(\bar{\beta}+\beta_0) \simeq 2\beta_0(\bar{\beta}-\beta_0)$

therefore:

$$i\partial_z \tilde{A} = -(\bar{\beta}(\omega) - \beta_0)\,\tilde{A}$$

$$\Leftrightarrow \quad \boxed{\partial_z \tilde{A} = i\,(\beta(\omega) + \delta\beta - \beta_0)\,\tilde{A}}$$

each frequency component within the pulse envelope acquires a phase-shift whose magnitude depends on both frequency ($\beta(\omega)$) and intensity (since $\delta\beta$ depends on $|E|^2$)

Let's now Taylor expand $\beta(\omega) = \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + o((\omega-\omega_0)^3)$

and remember the relation

$$(\partial_t)^\alpha \leftrightarrow [-i(\omega-\omega_0)]^\alpha \quad \text{since} \quad A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{A}(z,\omega-\omega_0)\,e^{-i(\omega-\omega_0)t}\,d\omega$$

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Then we have

$$\partial_z A = -\beta_1 \partial_t A - \frac{i}{2}\beta_2 \partial_{tt} A + i\delta\beta A$$

$$\Leftrightarrow \underbrace{(\partial_z + \beta_1 \partial_t) A}_{} = -\frac{i}{2}\beta_2 \partial_{tt} A + i\left(\gamma |A|^2 + \frac{i\alpha}{2}\right) A$$

we can do a change of variable $T = t - z/\beta_1$ and obtain:

$$\boxed{\partial_z A + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\alpha}{2} A = i\gamma |A|^2 A}$$

this is called the Nonlinear Schrödinger eq.

with the nonlinear parameter γ defined as $\gamma = \dfrac{n_2 \omega_0}{c A_{eff}}$

and $A_{eff} = \dfrac{\left(\iint |F(x,y)|^2 dx dy\right)^2}{\iint |F(x,y)|^4 dx dy}$

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