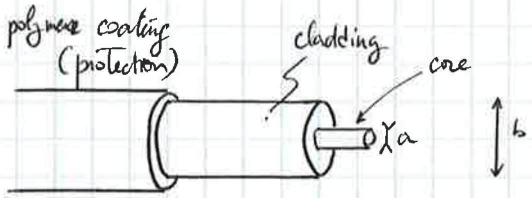
How to determine the mode profile in fiber ?



for telecom use : SiO_2 and core is doped with Ge → $n_{co} > n_{cl}$

or cladding doped with F or P → $n_{cl} < n_{co}$

for telecom
$$\begin{cases} \phi \simeq 9\,\mu m \\ \text{ext. } \phi \simeq 125\,\mu m \end{cases}$$

$\Delta n = n_{co} - n_{cl} \simeq 5 \times 10^{-3}$ this is really small !

Since we have cylindrical symmetry we can express $\vec{E}$ & $\vec{H}$ as

$$\begin{cases} \vec{E} = E(\vec{r}, \theta) \exp i(\omega t - \beta z) \\ \vec{H} = H(\vec{r}, \theta) \exp i(\omega t - \beta z) \end{cases}$$

And using the Maxwell's equation $\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$ and $\nabla \times H = \varepsilon \frac{\partial E}{\partial t}$ with $\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0 n^2$

and $\nabla = \left(\partial_r, \frac{1}{r}\partial_\theta, \frac{\partial}{\partial z}\right)$!

$$\Rightarrow \begin{cases} \frac{1}{r}\partial_\theta E_z + i\beta E_\theta = -i\omega\mu_0 H_r & (1) \\ -i\beta E_r - \frac{\partial E_z}{\partial r} = -i\omega\mu_0 H_\theta & (2) \\ \frac{1}{r}\partial_r(r E_\theta) - \frac{1}{r}\partial_\theta E_r = -i\omega\mu_0 H_z & (3) \end{cases}$$

and

$$\frac{1}{r}\partial_\theta H_z + i\beta H_\theta = i\omega\varepsilon_0 n^2 E_r \quad (4)$$

$$-i\beta H_r - \partial_r H_z = i\omega\varepsilon_0 n^2 E_\theta \quad (5)$$

$$\frac{1}{r}\partial_r(r H_\theta) - \frac{1}{r}\partial_\theta H_r = i\omega\mu_0 E_z \quad (6)$$

from (2) & (4) we can extract E_r by eliminating H_θ :

(2) $\quad -i\beta E_r - \partial_r E_z = -i\omega\mu_0 H_\theta \qquad \times\, i\beta$

(4) $\quad \frac{1}{r}\partial_\theta H_z + i\beta H_\theta = i\omega\varepsilon_0 n^2 E_r \qquad \times\, -i\omega\mu_0$

$$i\left(\beta\,\partial_r E_z + \frac{\omega\mu_0}{r}\partial_\theta H_z\right) = \left(\omega^2\varepsilon_0 n^2\mu_0 - \beta^2\right) E_r$$

$$\boxed{E_r = \frac{-i}{k^2 n^2 - \beta^2}\left(\beta\,\partial_r E_z + \frac{\omega\mu_0}{r}\partial_\theta H_z\right)} \quad (i)$$

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Similarly

$$\boxed{\begin{aligned} E_\theta &= \frac{-i}{k^2n^2-\beta^2}\left(\frac{\beta}{r}\partial_\theta E_z - \omega\mu_0\,\partial_r H_z\right) && \text{(ii)}\\ H_r &= \frac{-i}{k^2n^2-\beta^2}\left(\beta\,\partial_r H_z - \frac{\omega\varepsilon_0 n^2}{r}\partial_\theta E_z\right) && \text{(iii)}\\ H_\theta &= \frac{-i}{k^2n^2-\beta^2}\left(\frac{\beta}{r}\partial_\theta H_z + \omega\varepsilon_0 n^2\,\partial_r E_z\right) && \text{(iv)}\end{aligned}}$$

There exist 3 types of modes TE $(E_z=0)$; TM $(H_z=0)$ and Hybrid.

note moreover that we must have an azimuthal dependency (θ) and by symmetry this must be expressed as $\cos(n\theta+\phi)$ or $\sin(n\theta+\phi)$ where $n\in\mathbb{N}$.

And $\vec{E}$ and $\vec{H}$ must fulfill the Helmholtz eq. since they are propagating field $(\nabla^2+k^2)\begin{Bmatrix}E\\H\end{Bmatrix}=0$

with ∇^2 expressed in cylindrical coordinates: $\nabla^2 f = \frac{1}{r}\partial_r(r\,\partial_r f) + \frac{1}{r^2}\partial_{\theta\theta} f + \partial_{zz} f$

$$\text{(v)}\qquad \partial_{rr}\begin{Bmatrix}E_z\\H_z\end{Bmatrix} + \frac{1}{r}\partial_r\begin{Bmatrix}E_z\\H_z\end{Bmatrix} + \frac{1}{r^2}\partial_{\theta\theta}\begin{Bmatrix}E_z\\H_z\end{Bmatrix} + \left(k^2 n^2(r,\theta)-\beta^2\right)\begin{Bmatrix}E_z\\H_z\end{Bmatrix} = 0$$

$\downarrow$ by symmetry $n(r,\theta)\equiv n(r)$.

The idea here is to treat the case for TE mode and only give the main results for TE and hybrid modes.

TE mode $(E_z=0)$ $\rightarrow$ the propagation eq. only apply on H; and for symmetric reason we can express H_z as

$$H_z = \begin{Bmatrix} g(r)\\ h(r)\end{Bmatrix}\cos(n\theta+\phi) \qquad \begin{matrix} 0\leq r\leq a & \text{(core region)}\\ r>a & \text{(cladding region)}\end{matrix}$$

of course at the boundary core/cladding $(r=a)$ $g(a)=h(a)$!

$\hookrightarrow$ only apply on transverse component H_z & H_θ.

$\Rightarrow$ from (iv) : $H_\theta = \dfrac{-i\beta}{k^2 n^2 - \beta^2} \dfrac{1}{r} \partial_\theta H_z$

at $r=a$ this yields :

$$\frac{i\beta}{k^2 n_{co}^2 - \beta^2} \frac{n}{a} g(a) \sin(n\theta + \phi) = \frac{i\beta}{k^2 n_{cl}^2 - \beta^2} \frac{n}{a} h(a) \sin(n\theta + \phi)$$

and since $n_{co} \neq n_{cl}$, this can only be true if $n=0$!

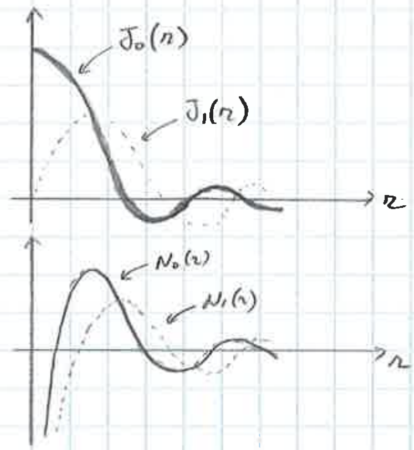
this also means that any component expressed as a function of $\partial_\theta H_z$ would be null : $E_r ; H_\theta$.

Finally for TE modes the propagation eq. (v) becomes:

$$\frac{d^2 H_z}{dr^2} + \frac{1}{r}\frac{dH_z}{dr} + \left(k^2 n^2(r) - \beta^2\right) H_z = 0 \qquad (6)$$

and the fields :

$$\begin{cases} E_\theta = \dfrac{i\omega\mu_0}{k^2 n^2(r) - \beta^2} \dfrac{dH_z}{dr} \\[2ex] H_r = \dfrac{-i\beta}{k^2 n^2 - \beta^2} \dfrac{dH_z}{dr} \\[2ex] E_r = 0 \\ H_\theta = 0 \end{cases}$$



eq. 6 has the form : $\dfrac{d^2 g}{dr^2} + \dfrac{1}{r}\dfrac{dg}{dr} + \alpha^2 g = 0$ with $\alpha^2 = k^2 n_{co}^2 - \beta^2$ in the core

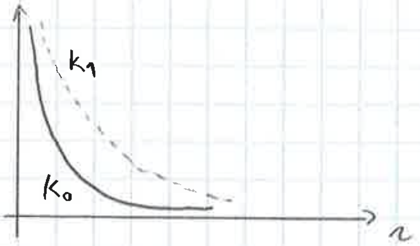
$\rightarrow$ solution as Bessel function . $\begin{cases} J_0(\alpha r) & \text{Bessel fct of } 0^{th} \text{ order} \\ N_0(\alpha r) & 0^{th} \text{ order Neumann function.} \end{cases}$

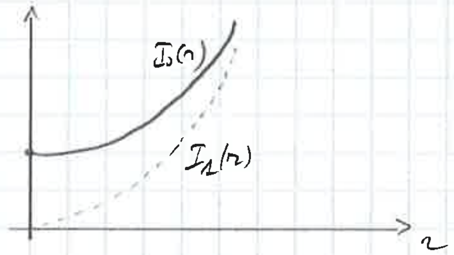
However since $N_0(\alpha r)$ diverges for $r=0$, this cannot be a solution.

In the cladding : $h'' + \frac{1}{r} h' - \gamma^2 h = 0$ with $\gamma^2 = \beta^2 - k^2 n_{cl}^2$

→ solutions $I_0(\gamma r)$ modified Bessel function of 1st kind) diverges for $r \to +\infty$!

$K_0(\gamma r)$ ———————— 2nd kind





Finally : $H_z = \begin{cases} A\, J_0(\alpha r) & \forall\ |r| \leq a \quad \text{core region} \\ B\, K_0(\gamma r) & \forall\ r > a \quad \text{cladding region} \end{cases}$

the continuity equations for H_z & E_θ at the boundary $r = a$ yield :

$$\begin{cases} A\, J_0(\alpha a) = B\, K_0(\gamma a) \\ \frac{A}{\alpha} J_0'(\alpha a) = -\frac{B}{\gamma} K_0'(\gamma a) \end{cases}$$

$$\Rightarrow \quad \frac{J_0'(\alpha a)}{\alpha\, J_0(\alpha a)} = -\frac{K_0'(\gamma a)}{\gamma\, K_0(\gamma a)}$$

we can introduce the dimensionless variable $U = \alpha a$ and $W = \gamma a$ then we obtain

$$\boxed{\frac{J_0'(U)}{U\, J_0(U)} = \frac{-K_0'(W)}{W\, K_0(W)}}$$

this is an eigenvalue pb.

Remember that $\left.\begin{array}{l} \alpha^2 = k^2 n_{co}^2 - \beta^2 \\ \gamma^2 = \beta^2 - k^2 n_{cl}^2 \end{array}\right\} \Rightarrow (\alpha a)^2 + (\gamma a)^2 = U^2 + W^2 = a^2 k^2 (n_{co}^2 - n_{cl}^2) = V^2$ only depends on the fiber!

$\Rightarrow$ strategy to evaluate the mode in the fiber:

(1) for a given (a, n_{co}, n_{cl}) solve the eigenvalue eq.

(2) this yields $H_z = \begin{cases} A\, J_0(\kappa r) \\ B\, K_0(\gamma r) \end{cases}$

(3) calculate E_θ & H_r:

in the core

$$\begin{cases} E_\theta = -i\omega\mu_0 \frac{a}{u} A\, J_1\left(\frac{u}{a} r\right) \\ H_r = i\beta \frac{a}{u} A\, J_1\left(\frac{u}{a} r\right) \\ H_z = A\, J_0\left(\frac{u}{a} r\right) \end{cases}$$

in the cladding

$$\begin{cases} E_\theta = i\omega\mu_0 \frac{a}{W} \frac{J_0(u)}{K_0(W)} \cdot A\, K_1\left(\frac{W}{a} r\right) \\ H_r = -i\beta \frac{a}{W} \frac{J_0(u)}{K_0(W)} A\, K_1\left(\frac{W}{a} r\right) \\ H_z = \frac{J_0(u)}{K_0(W)} A\, K_0\left(\frac{W}{a} r\right) \end{cases}$$

for TM modes and for hybrid modes, the strategy is the same but the eigenvalue eq. is somehow different.

- TM modes : $(H_z = 0)$

$$\boxed{\frac{J_1(u)}{u J_0(u)} = -\left(\frac{n_{cl}}{n_{co}}\right)^2 \frac{K_1(W)}{W K_0(W)}}$$

- hybrid modes $\begin{pmatrix} E_z \neq 0 \\ H_z \neq 0 \end{pmatrix}$

$$\boxed{\left[\frac{J_n'(u)}{u J_n(u)} + \frac{K_n'(W)}{W K_n(W)}\right]\left[n_{co}^2 \frac{J_n'(u)}{u J_n(u)} + n_{cl}^2 \frac{K_n'(W)}{W K_n(W)}\right] = \frac{\beta^2}{k^2}\left(\frac{1}{U^2} + \frac{1}{W^2}\right) n^2}$$