$\chi^{(3)}$ processes with pulses. NLO in optical fibers

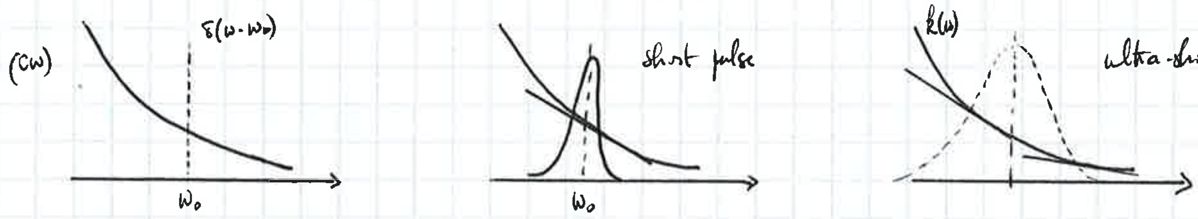
Historically the study of the propagation of pulses in fiber is motivated by telecommunication and the theory of information. Since light at a single frequency $E(z,t) = A \exp[i(\omega_0 t - k_0 z)]$ cannot carry information it is important to modulate the amplitude A. The amount of information depends on how fast $A(t)$ varies, which is simply linked with the spectral bandwidth of the input field such that the input field is

$$E(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{A(\omega-\omega_0)}_{\text{complex amplitude of each spectral component}} e^{i(\omega t - kz)} \, d\omega$$

$k = k(\omega) = \frac{n(\omega)\cdot\omega}{c}$ is the dispersion relation.

The more information we want to encode, the more rapidly $A(t)$ must vary and therefore the larger the spectral bandwidth of the initial pulse.

Depending of how broad is the spectrum of the initial signal the dispersion relation can be expanded with various order:



$$k = k_0 = \frac{n(\omega_0)\cdot\omega_0}{c} \qquad k = k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0) \qquad k = k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\left(\frac{d^2k}{d\omega^2}\right)_{\omega_0}(\omega-\omega_0)^2 + \dots$$

<u>for short pulse</u>: $k = k_0 + \left(\frac{dk}{d\omega}\right)_{\omega_0}(\omega-\omega_0) + o[(\omega-\omega_0)^2]$

$$E(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(\omega-\omega_0) \exp i\left\{\omega t - \left[k_0 + \left(\frac{dk}{d\omega}\right)(\omega-\omega_0)\right]z\right\} d\omega$$

$$= \frac{1}{2\pi} e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{+\infty} A(\Omega) \exp i\left[\Omega t - \left(\frac{dk}{d\omega}\right)\Omega z\right] d\omega$$

$$\boxed{E(z,t) = A\left(t - \frac{z}{v_G}\right) \exp i(\omega_0 t - k_0 z)}$$ with $v_G = 1/(dk/d\omega)_{\omega_0}$

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$\Rightarrow$ the information carried by the modulation of A is propagating at the group velocity defined by

$$\boxed{\frac{1}{v_G} = \left(\frac{dk}{d\omega}\right)_{\omega_0}}$$

<u>ultra-short pulses</u> it is clear from the picture that we can now define different group velocities. The fastest is at the frequency ω_1 and the slowest at ω_2. As the information (pulse) travels, the influence of these different group velocity yields a group delay:

$$\Delta\tau_G = z\left[\frac{1}{v_G^{(2)}} - \frac{1}{v_G^{(1)}}\right] = z\left[\left(\frac{dk}{d\omega}\right)_{\omega_2} - \left(\frac{dk}{d\omega}\right)_{\omega_1}\right]$$

and if $v_G^{(1)} \simeq v_G^{(2)}$ then

$$\Delta\tau_G = z\left[\frac{1}{v_G^{(1)}} + \frac{d}{d\omega}\left(\frac{1}{v_G}\right)_{\omega_1}(\omega_2-\omega_1) - \frac{1}{v_G^{(1)}(\omega_1)}\right] = z\,\frac{d}{d\omega}\left(\frac{dk}{d\omega}\right) = \frac{d^2k}{d\omega^2}$$

$$\Rightarrow \boxed{\Delta\tau_G = L\underbrace{\left(\frac{d^2k}{d\omega^2}\right)_{\omega_0}}_{\text{Group velocity dispersion.}}(\omega-\omega_0)}$$

(take care of units)

We see clearly that appears the term $\left(\frac{d^2k}{d\omega^2}\right)$ which was in the Taylor expansion for ultrashort pulses. This term was necessary for ultrashort pulses when the initial Taylor expansion was no longer valid. We could of course carry on like this with higher-order expansion term but this would not really give more information.

We already showed that the presence of $\chi^{(3)}$ induces an intensity-dependent refractive index $n = n_0 + n_2 I$

the "zero" indicates low power here.

Considering the relation between intensity and $|E|^2$ we could write the Kerr contribution as

$$n = n_0 + \bar{n}_2 |E|^2$$

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Since the wave number k is directly linked to the refractive index by

$$k = \frac{\omega \cdot n}{c}$$

$$\mapsto \quad k = \frac{\omega\, n_0(\omega)}{c} + \frac{\omega\, \bar{n}_2}{c}\, |A|^2 = k(\omega, |A|^2)$$

The most convenient way to derive a global equation which would describe the evolution of the envelope, taking into account the effects of dispersion [linear part of $k(\omega)$] and the Kerr-effect [non linear part of $k(\omega)$] is to Taylor expand $k(\omega, |E|^2)$ around ω_0, and around the electric field intensity $|A|^2$:

$$(1) \qquad k = k_0 + \left(\frac{dk}{d\omega}\right)(\omega-\omega_0) + \frac{1}{2}\left(\frac{d^2k}{d\omega^2}\right)(\omega-\omega_0)^2 + \underbrace{\frac{\partial k}{\partial |A|^2}}_{\mapsto \frac{\omega \cdot \bar{n}_2}{c}} \cdot |A|^2$$

Finally if we look at the Fourier transform by considering that a change $\Delta\omega = \omega - \omega_0$ will yield a change $\Delta k = (k - k_0)$

$$\tilde{E}(\Delta k, \Delta\omega) \propto \iint E(z,t)\, e^{-i(\Delta\omega t - \Delta k z)}\, dz\, dt$$

$$E(z,t) \propto \iint \tilde{E}(\Delta k, \Delta\omega)\, e^{+i(\Delta\omega t - \Delta k z)}\, d\omega\, dk$$

We can identify that $\quad k - k_0 \leftrightarrow +i\partial_z$

$$\omega - \omega_0 \leftrightarrow -i\partial_t$$

Let us now use the operator (eq. (1)) on the envelope of the input field:

$$(k - k_0)\, A = k'(\omega_0)(\omega-\omega_0) A + \frac{1}{2} k''(\omega_0)(\omega-\omega_0)^2 A + \frac{\partial k}{\partial |A|^2} \cdot |A|^2 A$$

$$\downarrow$$

$$i\partial_z A = -i\, k'(\omega_0) \frac{\partial}{\partial t} A - \frac{1}{2} k''(\omega_0)\, \partial_{tt} A + \frac{\omega \cdot \bar{n}_2}{c} |A|^2 \cdot A$$

$$\Rightarrow \quad i\left[\partial_z + k'\partial_t\right]A = -\frac{1}{2}k''\partial_{tt}A + \frac{\omega n_2}{c}|A|^2A \qquad (1)$$

Note that for small amplitude ($|A|^2A$ negleageable) and no high-order dispersion term, we only have

$i\left(\partial_z + k'\partial_t\right)A = 0$ which has a trivial solution that can be expressed by any function of the variable $(z - t/k')$, corresponding to the situation when the pulse propagates a the groupe-velocity w/o changing its shape.

A good practice is to introduce the change of variable $T = t - \frac{z}{v_g}$. This means that the observer is travelling at the same speed as the pulse. The eq. (1) then becomes:

$$\boxed{i\,\partial_z A = -\frac{1}{2}k''\partial_{TT}A + \frac{\omega n_2}{c}|A|^2A}$$

Non linear Schrödinger equation.

Note that so far we have not consider the fact that we were in a fiber, or any type of waveguide!

NLSE in optical fiber.

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NLS from Maxwell equations.

Maxwell $\nabla \times E = -\frac{\partial B}{\partial t}$ $\nabla . D = 0$ no charge, no loss. we are in an optical fiber (SiO_2)

$\nabla \times H = \frac{\partial D}{\partial t}$ $\nabla . B = 0$

⊕ material equations : $\begin{cases} D = \varepsilon_0 E + P \\ B = \mu_0 H \end{cases}$ (no magnetisation)

$$\nabla \times \nabla \times E = \nabla \times \left[-\mu_0 \frac{\partial}{\partial t} H\right] = -\mu_0 \frac{\partial}{\partial t}\left[\nabla \times H\right] = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} D\right) = -\mu_0 \varepsilon_0 \partial_{tt} E - \mu_0 \partial_{tt} P$$

$$\Rightarrow \nabla \times \nabla \times E = -\frac{1}{c^2}\partial_{tt} E - \mu_0 \frac{\partial^2 P}{\partial t^2} = \nabla(\nabla \cdot E) - \nabla^2 E = -\nabla^2 E \qquad (1)$$

$$P = P_{lin} + P_{NL} = \varepsilon_0 \left(\chi^{(1)} E + \chi^{(3)} E \vdots E \vdots E\right).$$

In general if the response is not instantaneous we need to use the frequency dependence of $\chi^{(i)}$. And since $\tilde{F}(\omega) \cdot \tilde{G}(\omega) \leftrightarrow \int g(t) f(t-\tau)\, d\tau$ the polarization, in its most general form is in fact:

$$P = \varepsilon_0 \int \chi^{(1)}(\vec{x},\tau)\, \vec{E}(\vec{x}, t-\tau)\, d\tau$$

(2)

$$+ \varepsilon_0 \iiint \chi^{(3)}(\vec{x}, t_1, t_2, t_3) \vdots E(\vec{x}, t-t_1)\, E(\vec{x}, t-t_2)\, E(\vec{x}, t-t_3)\, dt_1\, dt_2\, dt_3$$

which is valid in the electric dipole approx. and assuming that the medium response is <u>local</u>.

This is obviously hard to handle directly (1) & (2). The 1st idea is that the non linearity is a perturbation.

<u>Step 1</u> $P_{NL} = 0 \Rightarrow \nabla \times \nabla \times E = -\frac{1}{c^2}\partial_{tt} E - \mu_0 \varepsilon_0\, \partial_{tt}\left(\chi^{(1)} E\right)$

in the Fourier domain $\nabla \times \nabla \times \tilde{E} = \frac{\omega^2}{c^2}\tilde{E} + \frac{\omega^2}{c^2}\chi^{(1)}(\omega)\,\tilde{E}$

$\tilde{E}(\omega) = \int_{-\infty}^{+\infty} E(t)\, e^{i\omega t}\, dt$

$$\Rightarrow \nabla \times \nabla \times E = \frac{\omega^2}{c^2} \underbrace{\left[1 + \chi^{(1)}(\omega)\right]}_{\varepsilon(\omega)} \tilde{E} = -\nabla^2 E$$

we know that $\chi^{(1)}(\omega)$ can be complex and
$$\begin{cases} n(\omega) = 1 + \frac{1}{2} Re\left[\chi^{(1)}(\omega)\right] & \text{refractive index} \\ \alpha(\omega) = \frac{\omega}{nc} Im\left(\chi^{(1)}\right) \end{cases}$$

In case losses are small we obtain : $\boxed{\nabla^2 \tilde{E} + \frac{\omega^2}{c^2} n^2 \tilde{E} = 0}$ Helmholtz eq.

<u>General case:</u> let us consider a $\chi^{(3)}$ material. the NL polarization is :

$$P_{NL} = \varepsilon_0 \chi^{(3)} \vdots \vec{E}(\vec{r},t) \vec{E}(\vec{r},t) \vec{E}(\vec{r},t)$$

this is the Kerr effect and this should remain a perturbation.

$$\Leftrightarrow \varepsilon(\omega) = 1 + \chi^{(1)} + \varepsilon_{NL}$$

$$\varepsilon_{NL} = \frac{3}{4} \chi^{(3)} |E|^2 \quad \text{is intensity dep.}$$

As previously we can define quantities linked with $Re(\varepsilon_{NL})$ and $Im(\varepsilon_{NL})$

$$\rightarrow \tilde{n} = n_0 + n_2 |E|^2 \qquad \tilde{\alpha} = \alpha_0 + \alpha_2 |E|^2$$

$n_0 = 1 + \frac{1}{2} Re(\chi)$, $n_2 = \frac{3}{8} Re(\chi^{(3)})$

$\alpha_2 = \frac{3}{4} \frac{\omega_0}{c} Im(\chi^{(3)})$ is linked with the process of 2-photons absorption and is usually neglected.

In this case, the electric field defined by its spectrum.

$$\tilde{E}(\vec{r}, \omega - \omega_0) = \int_{-\infty}^{+\infty} E(\vec{r},t) \exp\left[i(\omega-\omega_0)t\right] dt \quad \text{is solution of the Helmholtz eq.}$$