Nonlinear Optics
Optics with $\chi^{(3)}$ materials

# 0.1 Electro-optical effects

## 0.1.1 Tensors and index ellipsoid

We have already discussed that since **D** and **E** are linked by a tensorial relationship $\mathbf{D} = \epsilon_0 \hat{\epsilon}_r \mathbf{E}$, where $\hat{\epsilon}_r$ is a tensor. We can rewrite this equation as

$$\forall i \in (x, y, z), \quad D_i = \epsilon_0 \sum_j \hat{\epsilon}_{r,ij} E_j \tag{1}$$

In the most general form the tensor $\hat{\epsilon}_r$ is

$$\hat{\epsilon}_r = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \cdots & \\ \vdots & \epsilon_2 & \\ & & \epsilon_3 \end{bmatrix} \tag{2}$$

where $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ are the susceptibility along the three eigenaxis of the tensor $\hat{\epsilon}_r$. We remind that we previously wrote from the tensor $\hat{\epsilon}_r$ the index ellipsoid

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \tag{3}$$

where $n_i = \sqrt{\epsilon_i}$. We also remind that depending on the values of the refractive indices $n_i$, the material can be characterized as

- *isotropic material* if $\epsilon_1 = \epsilon_2 = \epsilon_3$
- *uniaxial material* if $\epsilon_1 = \epsilon_2 \neq \epsilon_3$
- *biaxial material* if $\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$

## 0.1.2 Impermeability

In the case of electro-optical material the refractive of index can be modified when an external field (DC or not) is applied. Actually, such a modification of the refractive index will affect the index ellipsoid, but instead of dealing with ratio as $\left({}^1\!/\!{n_i^2}\right)$ it is actually convenient to define the inverse of the permittivity, which is called the impermeability $\hat{\eta}$ such that

$$\hat{\eta} \cdot \hat{\epsilon}_r = \mathbb{1} \;\Rightarrow\; \mathbf{E} = \frac{1}{\epsilon_0} \hat{\eta} \mathbf{D} \tag{4}$$

Using the impermeability tensor, the index ellipsoid is then written as

$$\eta_{ij}(E) x_i^2 = 1 \tag{5}$$

In this form, applying an external electric field will lead to a small variation of the refractive index, which can easily be treated as a perturbation of the impermeability tensor $\hat{\eta}$. Interestingly, such effect can be advantageously used to control either the intensity or the phase of the propagating radiation. Two cases are important:

- *Pockels effect*: the variation of the refractive index is linear with the applied electric field. It can only occur with *non-centro symmetric* crystal.

- *Kerr effect*: the variation of the refractive index is quadratic with the applied electric field.

The perturbation that are created yield a modification of the impermeability tensor

$$\eta_{ij}(E) = \eta_{ij}(0) + \sum_k \left(\frac{\partial \eta_{ij}}{\partial E_k}\right)_{E=0} E_k + \frac{1}{2}\sum_{k,\ell}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_\ell}\right) E_k E_\ell + \dots \tag{6}$$

It is common to redefine

$$r_{ijk} = \left(\frac{\partial \eta_{ij}}{\partial E_k}\right)_{E=0} \quad \text{and} \quad s_{ijk\ell} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_\ell}\right) E_k E_\ell \tag{7}$$

such that

$$\eta_{ij}(E) = \eta_{ij}(0) + \sum_{k=x,y,z} r_{ijk} E_k + \sum_{k,\ell=x,y,z} s_{ijk\ell} E_k E_\ell \tag{8}$$

Finally, in order to avoid the sum in the equation, it is important to use the Einstein notation

$$\sum_{k=x,y,z} r_{ijk} E_k \Rightarrow r_{ijk} E_k$$
$$\sum_{k,\ell=x,y,z} s_{ijk\ell} E_k E_\ell \Rightarrow s_{ijk\ell} E_k E_\ell$$

### 0.1.3 A bit of symmetry

Since $\hat{\epsilon}_r$ is a symmetric tensor, then $\hat{\eta}$ is also a symmetric tensor: $\eta_{ij} = \eta ji$:

$$\hat{\epsilon}_r = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{1} & \epsilon_{6} & \epsilon_{5} \\ \epsilon_{6} & \epsilon_{2} & \epsilon_{4} \\ \epsilon_{5} & \epsilon_{4} & \epsilon_{3} \end{bmatrix} \tag{10}$$

This $3 \times 3$ tensor can then be expressed as a vector of dimension 6. This is called the Voigt notation. Of course we can use this notation for either the susceptibility tensor or the impermeability tensor.

$$\hat{\epsilon}_r = \begin{pmatrix} \epsilon_1 &=& \epsilon_{11} \\ \epsilon_2 &=& \epsilon_{22} \\ \epsilon_3 &=& \epsilon_{33} \\ \epsilon_4 &=& \epsilon_{23} = \epsilon_{32} \\ \epsilon_5 &=& \epsilon_{13} = \epsilon_{31} \\ \epsilon_6 &=& \epsilon_{12} = \epsilon_{21} \end{pmatrix}; \quad \eta = \begin{pmatrix} \eta_1 &=& \eta_{11} \\ \eta_2 &=& \eta_{22} \\ \eta_3 &=& \eta_{33} \\ \eta_4 &=& \eta_{23} = \eta_{32} \\ \eta_5 &=& \eta_{13} = \eta_{31} \\ \eta_6 &=& \eta_{12} = \eta_{21} \end{pmatrix} \tag{11}$$

**Tensor for Pockels effect**

Similarly we can use the Voigt notation to reduce the [ $3 \times 3 \times 3 = 27$ elements ] tensor relative to the Pockels effect $r_{ijk}$ such that the impermeability tensor becomes:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix} (E) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix}_{(0)} + \begin{pmatrix} \Delta \left(\frac{1}{n^2}\right)_1 \\ \vdots \\ \vdots \\ \Delta \left(\frac{1}{n^2}\right)_6 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix}_{(0)} + \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & \\ r_{31} & & \\ r_{41} & & \\ r_{51} & & \\ r_{61} & & \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{12}$$

which is the very same equation as eq. (6).

**Tensor for Kerr effect**

As for the tensor relative to the Pockels effect, the tensor for the Kerr effects has symmetries than can be exploited in order to considerably reduced its dimension. Note that without taking into account those symmetries, the total dimension of that tensor is $3 \times 3 \times 3 \times 3 = 81$ elements!
First of all $s_{ijk\ell} = s_{jik\ell}$, but also since

$$\left(\frac{\partial \eta_{ij}}{\partial E_k \, \partial E_\ell}\right) \equiv \left(\frac{\partial \eta_{ij}}{\partial E_\ell \, \partial E_k}\right)$$

then $s_{ijk\ell} = s_{ij\ell k}$. Therefore the original tensor can be expressed by a $6 \times 6$ tensor:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix} (E) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix}_{(0)} + \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & \cdots & & & \\ s_{31} & \vdots & \ddots & & & \\ s_{41} & & & & & \\ s_{51} & & & & & \\ s_{61} & & & & & s_{66} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \\ 2E_yE_z \\ 2E_xE_z \\ 2E_xE_y \end{pmatrix} \tag{13}$$

Hopefully, we will be able to reduce these tensors even more...

## 0.2 The linear electro-optics effect

The linear electro-optics effect can only appear with non centro-symmetric crystal, which do not posses the inversion symmetry. We can use the very same argument that we use to prove that only non centro-symmetric crystal can exhibit a second order nonlinearity $\chi^{(2)}$. Let assume that we use a centro-symmetric crystal and apply a field such that it yields a change of refractive index $\Delta n_1 = sE$ where $s$ is a proportionality constant characteristic of the electro-optic effect. If we reverse the electric field the change in refractive index will be $\Delta n_2 = s(-E)$. Since the crystal posses the inversion symmetry property, we must have $\Delta n_1 = \Delta n_2$ which occurs if and only if $s = 0$. Therefore centro-symmetric crystal can only exhibit quadratic electro-optic effect. Non centro-symmetric crystal on the other hand can also present linear electro-optic effect.

### 0.2.1 Example: KDP

KDP ($KH_2PO_4$) is a uniaxial crystal. As for other uniaxial crystal the susceptibility tensor (and therefore the impermeability) tensor presents two distinct eigenvalues:

$$\hat{\eta} = \begin{pmatrix} \frac{1}{n_o^2} & 0 & 0 \\ 0 & \frac{1}{n_o^2} & 0 \\ 0 & 0 & \frac{1}{n_e^2} \end{pmatrix} \tag{14}$$

And the index ellipsoid is described by the equation:

$$\frac{x^2+y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \tag{15}$$

In the presence of an external electric field there is a modification of the refractive indices according to

$$\begin{pmatrix} \Delta\left(\frac{1}{n^2}\right)_1 \\ \vdots \\ \vdots \\ \Delta\left(\frac{1}{n^2}\right)_6 \end{pmatrix} = r_{ij} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{16}$$

where in the particular case of KDP, due to the symmetry properties of the crystal yield the electro-optic tensor has the following form

$$r_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{pmatrix} \tag{17}$$

#### Electric field along the $z$-direction

Considering that the applied field is $\mathbf{E} = E_z\,\mathbf{e}_z$, we can readily compute the impermeability tensor $\eta$ by adding eq. (14) and eq. (16) to get

$$\hat{\eta}(E) = \begin{pmatrix} \frac{1}{n_o^2} & r_{63}\,E_z & 0 \\ r_{63}\,E_z & \frac{1}{n_o^2} & 0 \\ 0 & 0 & \frac{1}{n_e^2} \end{pmatrix} \tag{18}$$

From this impermeability tensor we can calculate its eigenvalues and eigenvectors. As previously the eigenvectors will represents the principal axis of the crystal and the eigenvalues the refractive indices along these axis.

**Eigenvalues** The eigenvalues are calculated by solving the equation $det\,(\hat{\eta} - \lambda \mathbb{1}) = 0$. This is simply given by

$$\left(\frac{1}{n_e^2} - \lambda\right)\left[\left(\frac{1}{n_o^2} - \lambda\right)^2 - (r_{63}\ E_z)^2\right] =$$
$$\left(\frac{1}{n_e^2} - \lambda\right)\left(\frac{1}{n_o^2} + r_{63}\ E_z - \lambda\right)\left(\frac{1}{n_o^2} - r_{63}\ E_z - \lambda\right) = 0 \tag{19}$$

It is clear that the eigenvalues of the impermeability tensor are

$$\begin{cases} \lambda_1 &= {}^1\!/n_o^2 + r_{63}E_z \\ \lambda_2 &= {}^1\!/n_o^2 - r_{63}E_z \\ \lambda_3 &= {}^1\!/n_e^2 \end{cases} \tag{20}$$

The change of refractive index can be calculated by

$$\frac{1}{n_u^2} = \frac{1}{n_o^2} + d\left(\frac{1}{n_o^2}\right) = \frac{1}{n_o^2} + r_{63}\ E_z \tag{21}$$

If we assume that $r_{63}\ E_z \ll {}^1\!/n_o^2$ and calculating the differentiation $d\left({}^1\!/n^2\right) = \left({}^{-2}\!/n^3\right) dn$ we readily find the change of refractive index:

$$dn = \frac{-n_o^3}{2} r_{63}\ E_z \tag{22}$$

and therefore the new refractive indices (along **u** and **v**) are

$$n_{\mathbf{u}} = n_0 - \frac{1}{2} n_o^3 r_{63}\ E_z \tag{23}$$
$$n_{\mathbf{v}} = n_0 + \frac{1}{2} n_o^3 r_{63}\ E_z \tag{24}$$

**Eigenvectors** The eigenvector corresponding to the eigenvalues are

$$\mathbf{u}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix};\quad \mathbf{v}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};\quad \mathbf{w}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{25}$$

the z-axis is not changed but the $x-$ (resp. y$-$) direction are rotated by $45^o$. Finally we have the new ellisoid of indices which can be mathematically described by

$$\left(\frac{1}{n_o^2} + r_{63}\ E_z\right) u^2 + \left(\frac{1}{n_o^2} - r_{63}\ E_z\right) v^2 + \frac{1}{n_e^2} z^2 = 1 \tag{26}$$

is plotted on Fig. 1. Note that we could write this as

$$\frac{u^2}{n_u^2} + \frac{v^2}{n_v^2} + \frac{z^2}{n_e^2} = 1$$

In the presence of an external electric field, the uniaxial crystal of KDP is now a biaxial crystal!

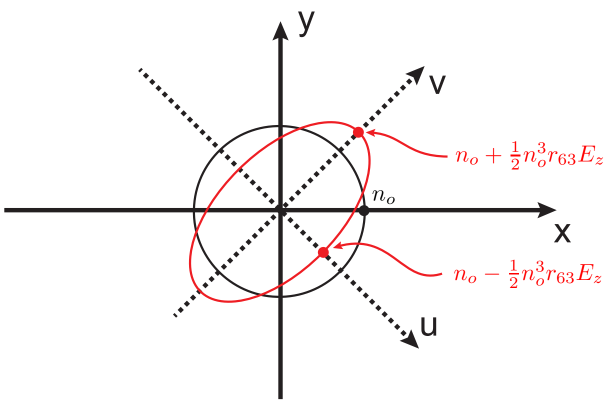


Figure 1: Index ellipsoid for KDP without any external field (black) and with an external field applied along the $z-$direction.

### 0.2.2 Example: Li-Niobate

The lithium niobate ($LiNbO_3$) is another important uniaxial crystal used for its electro-optical properties. It belongs to a different group point symmetry than the KDP crystal ($C_{3v}$). Due to this symmetry group the linear electro-optic tensor has the following form:

$$r_{ij} = \begin{pmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{pmatrix} \tag{27}$$

As previously let us assume that the external field is along the optic axis of the crystal. The equation of the index ellipsoid are in that case:

$$\left(\frac{1}{n_o^2 + r_{13}E}\right) x^2 + \left(\frac{1}{n_o^2 + r_{13}E}\right) y^2 + \left(\frac{1}{n_e^2 + r_{33}E}\right) z^2 = 1 \tag{28}$$

By contrast with the case of KDP, the applied field does not lead to a change of orientation of the principal axis of the crystal! However the length of the semiaxis of the ellipsoid are modified according to:

$$\begin{cases} n_x &= n_o - \frac{1}{2}n_0^3 r_{13} E \\ n_y &= n_o - \frac{1}{2}n_0^3 r_{13} E \\ n_z &= n_e - \frac{1}{2}n_e^3 r_{33} E \end{cases} \tag{29}$$

Let us now consider that the external field is applied along the $x-$axs of the crystal. The induced birefringence is then given by

$$n_z - n_y = (n_e - n_o) - \frac{1}{2}\left(n_e^3 r_{33} - n_o^3 r_{13}\right) E \tag{30}$$

### 0.2.3 General case

In general an external electric field causes a modification of the refractive indices and even the orientation of the principal axis of the crystal. This appears as mixed term in the equation of the ellipsoid of indices. In its most general formulation the equation of the ellipsoid of indices is

$$\left(\frac{1}{n_x^2} + r_{1k}E_k\right) x^2 + \left(\frac{1}{n_y^2} + r_{2k}E_k\right) y^2 + \left(\frac{1}{n_z^2} + r_{3k}E_k\right) z^2 + 2yz\ r_{4k}E_k + 2xz\ r_{5k}E_k + 2xy\ r_{6k}E_k = 0 \quad (31)$$

where the summation convention has to used. Namely $k$ represents the three coordinates $x, y, z$ so that $k = 1, 2, 3$ and $a_k b_k = \sum_{k=1}^{3} a_k b_k$.

## 0.2.4 Applications of the linear electro-optic effect

As we saw previously the application of an external electric field can change significantly the optical properties of the crystal. These effects can be efficiently used to modify and control the propagating light, in particular by affecting its polarization state. It is also possible to use those crystal to encode information on the traveling field either by phase or amplitude modulation. We are looking here at a few possible applications of the linear electro-optical effect.

### The Pockels cell

The way to use an electro-optic crystal is illustrated in its more basic way on Fig. 2. The main idea is to send a polarized beam onto the crystal. As we saw previously the application of an external static field yield a change of the refractive indices and possibly even modify the nature of the crystal[1].

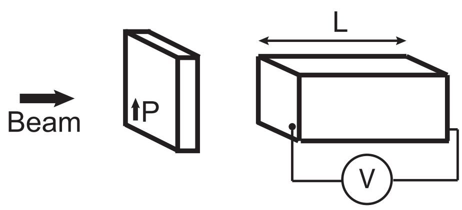


Figure 2: Principle of a Pockel cell. A polarizer indicated by the plate $P$ is placed before the electro-optic crystal.

### Induced birefringence and *retardation*

Let us consider a KDP crystal. The application of an external field along the z-axis of a KDP crystal modifies the orientation of its principal axis. In particular in the plane $z = 0$ the axis are rotated by $\pi/4$ (see Section. 0.2.1 and Fig. 1) . Let us assume that we use a beam polarized along the $x-$axis of the crystal. In the new referential ($\mathbf{u}$, $\mathbf{v}$) the field decomposes along both axis:

$$E_u = A \exp\left[i\left(\omega t - k_u z\right)\right] \quad (32a)$$

$$E_v = A \exp\left[i\left(\omega t - k_v z\right)\right] \quad (32b)$$

(32c)

[1]An uniaxial KDP crystal can turn into a biaxial crystal under the presence of an external field.

and at the end of the crystal, which has a length $\ell$:

$$E_u(\ell) = A\exp\left\{i\left[\omega t - \frac{\omega}{c}\left(n_o - \frac{1}{2}r_{63}n_o^3 E_z\right)\ell\right]\right\} \tag{33a}$$

$$E_v(\ell) = A\exp\left\{i\left[\omega t - \frac{\omega}{c}\left(n_o + \frac{1}{2}r_{63}n_o^3 E_z\right)\ell\right]\right\} \tag{33b}$$

As we see from eq. (33), as the field propagates both component acquire a different phase leading to a net phase-difference

$$\Delta\phi = \phi_u - \phi_v = k\cdot\ell\cdot\Delta n = \frac{\omega}{c}n_o^3 r_{63}E_z\ell = \frac{\omega}{c}n_o^3 r_{63}V \tag{34}$$

where the difference of potential $V = E\cdot\ell$. This net phase difference is called the *retardation*.

**half-wave voltage**

For a KDP crystal the retardation induced by the external electric field applied along the $z-$axis is given at eq. (34)

$$\Delta\phi == \frac{\omega}{c}n_o^3 r_{63}E_z\ell = \frac{2\pi}{\lambda}n_o^3 r_{63}V$$

which obviously depends not only on the length of the crystal $\ell$ but also on the strength of the applied difference of potential $V$. Using a field polarized along the $x-$axis (Fig. 1) of the crystal the field is described by the eq. (32), which we rewrite in a simpler form as

$$E_u = A\cos\omega t \tag{35a}$$

$$E_v = A\cos(\omega t - \Delta\phi) \tag{35b}$$

Depending on the value of the retardation $\Delta\phi$ the output beam can have various states of polarization. As shown on Fig. **??** there is a particular case for $\Delta\phi = \pi$. For this case

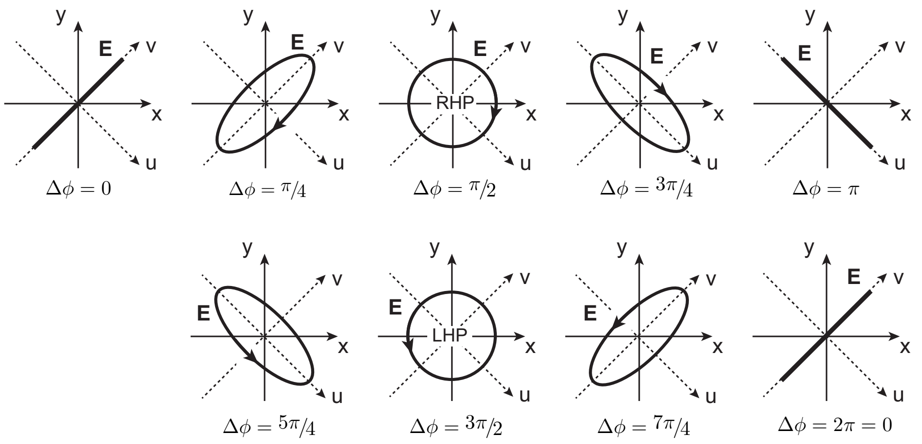


Figure 3: States of polarisation at the output of the electro-optical crystal according to the applied retardation $\Delta\phi$.

the field is linearly polarized (as initially) but is not along the $\mathbf{u}-$axis.

In the particular case of $\Delta\phi = \pi$ the crystal acts as an half-wave plate[2]. Since the induced phase is equal to $\pi$ we call this particular voltage the *half-wave voltage* $V_\pi$:

$$V_\pi = \frac{\lambda}{2n_o^3 r_{63}} \tag{36}$$

[2]We remind that a half-wave plate placed at an angle $\alpha$ with respect to the optics axis rotates a linear polarization by an angle equal to $2\alpha$.

In the particular case of KDP $r_{63} = 10.6 \times 10^{-12}\ m/V$ and $n_o = 1.47$ at $\lambda = 632$ nm. The corresponding half-wave voltage is then (eq. (36)) $V_\pi = 9.3$ kV. This is actually really not a small tension! The speed for modulating such a tension is limited to a few kHz. Using the expression for the half-wave voltage we can express the retardation in a general form

$$\Delta\phi = \pi \frac{V}{V_\pi} \tag{37}$$

**Use of the linear electro-optic effect for amplitude modulation**

As shown by the eq. (33) the application of an external electric field yields a birefringence, which causes the polarization to evolve along the crystal. We previously saw that in the case of beam polarized linearly along the $x-$axis, the application of a voltage equal to $V_\pi$ would lead to a beam linearly polarized along the $y-$axis. Suppose now that we place a polarizer after the crystal. The axis of the polarizer is along the $y-$axis. In the presence of the external electric field, the polarisation of the input beam has rotated and is now aligned with the axis of the polarizer: the beam goes through without any loss. On the other hand when the external electric field is off the linear polarisation of the input beam remains along the $x-$axis and is $90^o$ rotated with respect to the polarizer: the beam is blocked.

In a more general way we can use a polarizer in combination with an electro-optical crystal in order to modulated the amplitude of a beam. A typical arrangement for such scheme is presented on Fig. 4. For this scheme the input polarization is set along the $x-$axis of the crystal.

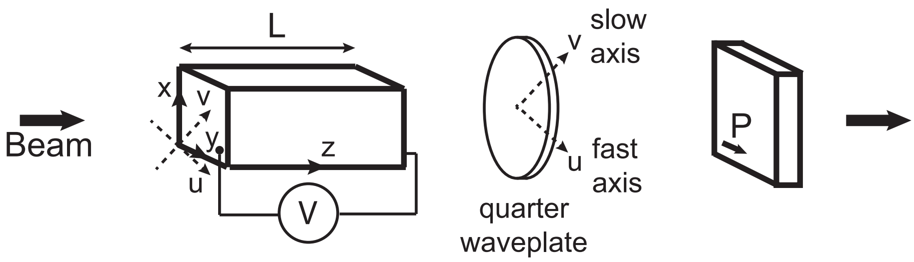


Figure 4: Typical scheme for amplitude modulation

Since the input polarisation is at $45^o$ with respect to the principal axis of the crystal (in the presence of an external electric field) the projection of the input field along the principal axis of the crystal are

$$E_u = A\cos\omega t \tag{38a}$$
$$E_v = A\cos\omega t \tag{38b}$$

which are in complex notation

$$E_u = A \tag{39a}$$
$$E_v = A \tag{39b}$$

Therefore the incident intensity is $I = |E_u|^2 + |E_v|^2 = 2A^2$. At the output of the scheme we have then

$$E_u = A \tag{40a}$$
$$E_v = Ae^{-i\Delta\phi_{tot.}} \tag{40b}$$

where the total phase shift includes the electrically-induced phase shift and the phase introduced by the quarter wave-plate:

$$\Delta\phi_{tot.} = \Delta\phi_{(EO-crystal)} + \Delta\phi_{(quarter-WP)} = \Delta\phi_{(EO-crystal)} + \frac{\pi}{2} \tag{41}$$

At the output of the scheme the $y-$component of the electric field is then

$$E_y = \frac{2}{\sqrt{2}}\left(e^{-i\Delta\phi_{tot.}} - 1\right) \tag{42}$$

and therefore the intensity

$$\begin{aligned} I_{output} \propto E_y E_y^* &= \frac{A^2}{2}\left(e^{-i\Delta\phi_{tot.}} - 1\right)\left(e^{+i\Delta\phi_{tot.}} - 1\right) \\ &= \frac{A^2}{2}\left[4\sin^2\left(\frac{\Delta\phi_{tot.}}{2}\right)\right] \end{aligned} \tag{43}$$

Considering the intensity of the input beam we have

$$\frac{I_{output}}{I_{input}} = \sin^2\left(\frac{\Delta\phi_{tot.}}{2}\right) = \sin^2\left(\frac{\pi}{2}\frac{V}{V_\pi}\right) \tag{44}$$

As we can see from Fig. 5, if we work close to half of the half-wave voltage the modulation of the intensity (eq. (44) is almost linear and therefore it is possible to modulate the intensity (and therefore the amplitude) of the beam without any deformation of the signal.

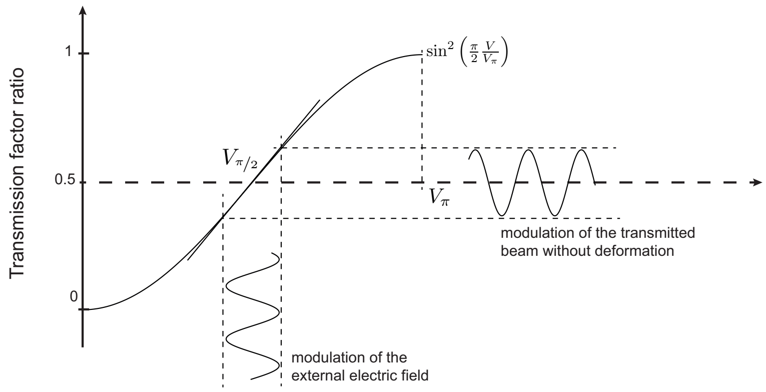


Figure 5: Use of the linear electro-optical effect for amplitude modulation. [Adapted from A. Yariv and P. Yeh, Optical Waves in Crystals - Wiley Interscience(1984)]

### Use of the linear electro-optic effect for phase modulation

Let us consider now that the input beam is polarized along the $u-$axis of the electro-optical crystal. In this case the component along the $v-$axis of the crystal is null. The application an the external electric field does not modify the polarisation state of the beam but induces a change of the phase following

$$\Delta\phi_u = \frac{-\omega}{c}\Delta n_u \ell = \frac{\omega n_o^3 r_{63}}{2c} E_z \ell \tag{45}$$

If the bias field is sinusoidal ($E_z = E_m \sin\omega_m t$), then an incident field assumed to be a plane wave (i.e. $E_{in} = A\cos\omega t$), will propagate through the crystal acquiring a sinusoidal phase. At the output plane the electric field associated with the light beam will be

$$E_{out} = A\cos\left[\omega t - \frac{\omega}{c}\left(n_o - \frac{n_o^3}{2} r_{63} E_m \sin\omega_m t\right)\ell\right] \tag{46}$$

And if we drop the constant factor we have the expression of the output field:

$$E_{out} = A\cos(\omega t + \delta \sin \omega_m t) \tag{47}$$

where $\delta$ is referred in literature as the phase modulation index and its expression is given by the following relation:

$$\delta = \frac{\omega n_o^3 r_{63} E_m \ell}{2\lambda} = \frac{\pi n_o^3 r_{63} E_m \ell}{\lambda} \tag{48}$$

The light beam then experiences phase modulation because of its propagation inside the crystal in such a configuration. The depth of the phase modulation is represented by the phase modulation index $\delta$. A scheme of an electro-optic phase modulator is shown in Fig. 6.

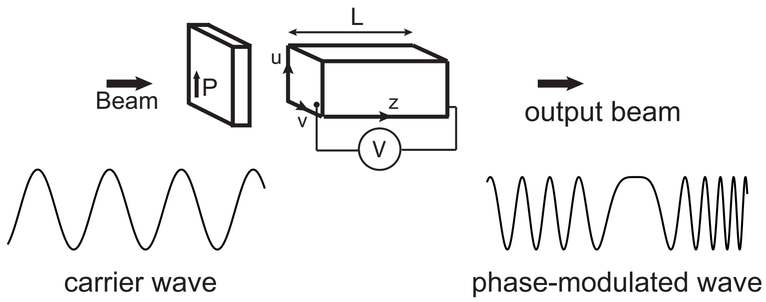


Figure 6: typical setup for phase-modulation of a optical beam

# .1 Electro-optic tensor depending of symmetry

There exists 32 different classes of crystal. For each class the susceptibility tensor has different vanishing coefficients [Boyd].

## .1.1 Triclinic symmetry

### Class 1

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{bmatrix}$$

## .1.2 Monoclinic symmetry

### class 2

$$axis \parallel \mathbf{y} \begin{bmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \\ r_{41} & 0 & r_{43} \\ 0 & r_{52} & 0 \\ r_{61} & 0 & r_{63} \end{bmatrix} \qquad axis \parallel \mathbf{z} \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \\ r_{41} & r_{42} & 0 \\ r_{51} & r_{52} & 0 \\ 0 & 0 & r_{63} \end{bmatrix}$$

### class m

$$m \perp \mathbf{y} \begin{bmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & r_{53} \\ 0 & r_{62} & 0 \end{bmatrix} \qquad m \perp \mathbf{z} \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \\ 0 & 0 & r_{43} \\ 0 & 0 & r_{53} \\ r_{61} & r_{62} & 0 \end{bmatrix}$$

## .1.3 Orthorhombic symmetry

### 222

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{52} & 0 \\ 0 & 0 & r_{63} \end{bmatrix}$$

### 2mm

$$\begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ r_{41} & r_{51} & 0 \\ r_{51} & -r_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### .1.4 Tetragonal symmetry

#### class 4 and $\bar{4}$

$$\text{class } 4 \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ r_{41} & r_{51} & 0 \\ r_{51} & -r_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{class } \bar{4} \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & -r_{13} \\ 0 & 0 & 0 \\ r_{41} & -r_{51} & 0 \\ r_{51} & r_{41} & 0 \\ 0 & 0 & r_{63} \end{bmatrix}$$

#### class 422 and 4mm

$$\text{class } 422 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & -r_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{class } 4mm \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

#### class $\bar{4}$2m

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{bmatrix}$$

| crystal | $r_{ij}$ $\times 10^{-22}$m.V$^{-1}$ |
|---|---|
| KDP | $r_{41} = 8.77$<br>$r_{63} = 10.3$ |
| KD*P | $r_{41} = 8.8$<br>$r_{63} = 26.8$ |
| ADP | $r_{41} = 23.76$<br>$r_{63} = 8.56$ |
| AD*P | $r_{41} = 40$<br>$r_{63} = 10$ |

### .1.5 Cubic symmetry

#### class $\bar{4}$3m

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{41} \end{bmatrix}$$

**class 432** $\Longrightarrow$ actually all coefficients vanish for crystal with this symmetry.

## .1.6 Trigonal symmetry

### class 3 and 32

$$\text{class 3}\quad \begin{bmatrix} r_{11} & -r_{22} & r_{13} \\ -r_{11} & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ r_{41} & r_{51} & 0 \\ r_{51} & -r_{41} & 0 \\ -r_{22} & -r_{11} & 0 \end{bmatrix} \qquad \text{class 32}\quad \begin{bmatrix} r_{11} & 0 & 0 \\ -r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & -r_{41} & 0 \\ 0 & -r_{11} & 0 \end{bmatrix}$$

### class 3m

$$m \perp \mathbf{x}\quad \begin{bmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{bmatrix} \qquad m \perp \mathbf{y}\quad \begin{bmatrix} r_{11} & 0 & r_{13} \\ -r_{11} & 0 & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ 0 & -r_{11} & 0 \end{bmatrix}$$

| crystal | $r_{ij}$ $\times 10^{-22}$m.V$^{-1}$ |
|---|---|
| $LiNbO_3$ | $r_{13} = 9.6$<br>$r_{22} = 6.8$<br>$r_{33} = 30.9$<br>$r_{51} = 31.6$ |

## .1.7 Hexagonal symmetry

### class 6, 6mm, 622 and $\bar{6}$

$$\text{class 6}\quad \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{33} \\ r_{41} & r_{51} & 0 \\ r_{51} & -r_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{class 6mm}\quad \begin{bmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{class 622}\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ r_{51} & -r_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{class } \bar{6}\quad \begin{bmatrix} r_{11} & -r_{22} & 0 \\ -r_{11} & r_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -r_{22} & -r_{11} & 0 \end{bmatrix}$$

### class $\bar{6}$m2

$$m \perp \mathbf{x}\quad \begin{bmatrix} 0 & -r_{22} & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -r_{22} & 0 & 0 \end{bmatrix} \qquad m \perp \mathbf{y}\quad \begin{bmatrix} r_{11} & 0 & 0 \\ -r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_{11} & 0 \end{bmatrix}$$