Modern Optics: Advanced optics

Maxwell equations and Poynting vector

Exercises' sheet No 1 Oct. 2017

Exercise 1 *General wave equation*

1. Demonstrate that for an isotropic non perfectly transparent dielectric material the wave equation for the electric vector $\mathbf{E}$ is given by

$$\triangle \mathbf{E} + \nabla \left[\mathbf{E} \cdot \nabla \left(\ln \epsilon \right) \right] - \frac{\epsilon \mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \mu \sigma \frac{\partial \mathbf{E}}{\partial t} + \left[\nabla \left(\ln \mu \right) \right] \times \left(\nabla \times \mathbf{E} \right) = 0 \quad (1)$$

   where ϵ (permitivity), μ (permeability) and σ (conductivity) are independent of time, but not independent of the position $\mathbf{r}$.

2. Derive a similar equation for the magnetic field $\mathbf{H}$.
3. What is the condition for both equations to be symmetrical for a replacement $\mathbf{E} \rightarrow \mathbf{H}$ and $\epsilon \rightarrow \mu$?

Exercise 2 *Poynting vector theorem*

For this exercise we are considering a material that is not a dielectric. This means that $\rho \neq 0$. On the other hand, for simplicity we assume that $\epsilon = 1$ and $\mu = 1$.

1. We remind that conservation of energy can be described as a function of time by using the Poynting vector theorem is

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \quad (1)$$

   where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector and u is the density.
   Show that in this expression $-\mathbf{J} \cdot \mathbf{E}$ represents the temporal change of the work of the electric field on the material.

2. Similarly we can derive an expression for the change of momentum of the material under the action of the field acting on a charged medium. Show that this can be written in a general form as

$$\frac{dP_{\text{mech.}}}{dt} = \int_V \left(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \right) dV \quad (2)$$

3. Use the electric and magnetic fields to express eq. (2) as

$$\frac{dP_{\text{mech.}}}{dt} + \frac{d}{dt} \int_V \epsilon_0 \left(\mathbf{E} \times \mathbf{B} \right) dV \quad (3)$$
$$= \epsilon_0 \int_V \left[\mathbf{E} \left(\nabla \cdot \mathbf{E} \right) - \mathbf{E} \times \left(\nabla \times \mathbf{E} \right) + c^2 \mathbf{B} \left(\nabla \cdot \mathbf{B} \right) - c^2 \mathbf{B} \times \left(\nabla \times \mathbf{B} \right) \right] dV$$

4. Show that for any vector $\mathbf{A}$ we can write the component along $\hat{\mathbf{e}}_\alpha$ as

$$\left[\mathbf{A} \left(\nabla \cdot \mathbf{A} \right) - \mathbf{A} \times \left(\nabla \times \mathbf{A} \right) \right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} \left(A_\alpha A_\beta - \frac{1}{2} \mathbf{A} \cdot \mathbf{A} \ \delta_{\alpha\beta} \right) \quad (4)$$

   where $\delta_{\alpha\beta}$ is the Kronecker function, and A_i is the component along $\hat{\mathbf{e}}_i$.

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5. Use eq. (4) to rewrite the eq. (2) as

$$\left[\frac{d}{dt}\left(P_{\text{mech.}} + P_{\text{field}}\right)\right]_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha,\beta} dV \tag{5}$$

where the momentum of the field is expressed as

$$P_{\text{field}} = \epsilon_0 \int_V \mathbf{E} \times \mathbf{B}\ dV = \mu_0 \epsilon_0 \int_V \underbrace{\mathbf{E} \times \mathbf{H}}_{S}\ dV \tag{6}$$

and

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}\right)\right] \tag{7}$$

is called the *Maxwell stress tensor*.

6. Use the divergence theorem to derive the conservation of momentum [1]

$$\frac{d}{dt}\left(P_{\text{mech.}} + P_{\text{field}}\right) = \oint_S \sum_\beta T_{\alpha\beta} n_\beta dS \tag{8}$$

1. This was the hard part...