# Advanced laser
# Polarization effects - Jones' formalism

## 0.1 Introduction

### 0.1.1 Maxwell's equations

Starting from the Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1b}$$

$$\nabla . \mathbf{D} = \rho \tag{1c}$$

$$\nabla . \mathbf{B} = 0 \tag{1d}$$

associated with the equations for the material:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \tag{2a}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{2b}$$

for a dielectric ($\rho = 0$) and non-magnetic medium ($\mathbf{M} = 0$), using the algebraic equation $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla . A) - \nabla^2 \mathbf{A}$ applied to eq. (1a), we derive the equation for the propagation of the electric field:

$$\left[\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = \frac{1}{\epsilon_0 c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{3}$$

This is a <u>vectorial</u> equation for which the atomic polarization plays the role of a source term, acting on the electric field.

### 0.1.2 Atomic polarization

When an electric field is applied onto a material, it will modify the cloud of electrons around each atom that constitute this material. Such displacement of the electron cloud induces a dipole moment and in the linear case, each contribution (microscopic effect) will add to generate a macroscopic polarization $\mathbf{P}$.

$$\mathbf{E} \ \rightarrow \ \mathbf{p} \ \rightarrow \mathbf{P} = \sum_i \mathbf{p}_i \ \rightarrow \mathbf{E}$$

#### isotropic medium

From the induced polarization, we can write the displacement $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi \, \mathbf{E}$. Inserting the polarization into the equation of propagation (Eq. (3)) we have

$$\left[\nabla^2 - \frac{1+\chi}{c^2} \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = 0 \tag{4}$$

and the velocity of the wave is $c/(\sqrt{1+\chi}) = c/n$.

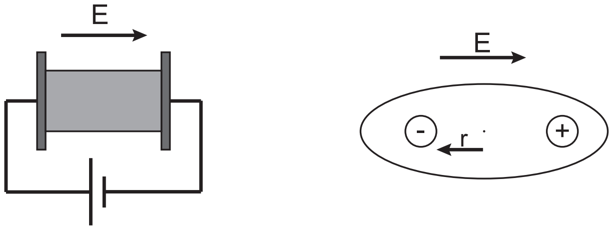



Figure 1: Induction of the atomic polarization by the electric field

**dispersion**

In the Lorentz model, the electron is bound to the atom, such that the action of the electric field acts like a driving force onto a harmonic oscillator. In this model, the motion of the electron can be described as

$$\ddot{r} + \gamma\dot{r} + \omega_0^2 r = \frac{-qE}{m_e} \tag{5}$$

where $r$ is the distance to the center of charge, $\gamma$ is a damping coefficient, $\omega_0$ the resonance frequency of the atom, $m_e$ the mass of the electron and $q$ its charge. The solution of the Eq. (5) is a superposition of a transient regime (eq. without driving force) with a permanent solution, that follows the driving force. We are only interesting by the permanent regime, and since the incident electric field is $E = E_0 \exp(j\omega t)$, we are seeking solution of the form $r = r_0 \exp(j\omega t)$. Substituting this into Eq. (5), we obtain:

$$r_0 = \frac{-q}{m_e\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)} E \tag{6}$$

And since the induced microscopic polarization is given by $p = (-q)r$, we have the macroscopic polarization

$$P = N(-q)r = \frac{Nq^2}{m_e} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} E \tag{7}$$

Since $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1 + \chi^{(1)})\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E}$, we see from eq. (7) than we can get the linear susceptibility $\chi^{(1)}$ or the susceptibility $\epsilon_r$. We can then use the relation

$$\epsilon_r = \left(n + \frac{i\alpha c}{2\omega}\right)^2 \tag{8}$$

where $n$ is the refractive index (linked to the real part of the susceptibility $\chi^{(1)}$) and $\alpha$ the absorption, which is linked with the imaginary part of $\chi^{(1)}$. Note that there exists many resonances in the system. Fig. 2 shows the evolution of the real part and the imaginary part of $\chi$ as a function of $\omega$.

**vectorial characteristics of E and B**

For a wave described by $E = E_0 e^{j(\omega t - \mathbf{k}.\mathbf{r})}$ and $B = B_0 e^{j(\omega t - \mathbf{k}.\mathbf{r})}$, propagating in a medium without charge nor current ($\rho = 0$, $\mathbf{J} = 0$, the laws of Faraday and Ampère can be rewrite as

$$-i\mathbf{k} \times \mathbf{E} = -i\omega\mathbf{B} \tag{9}$$
$$-i\mathbf{k} \times \mathbf{B} = -i\mu_0\omega\mathbf{D} \tag{10}$$

and in the case of a linear isotropic medium $\mathbf{D} = \epsilon_0(1 + \chi)\mathbf{E} \propto \mathbf{E}$, therefore $\mathbf{E} \perp \mathbf{k}$. If $z$ is the direction of propagation (along $\mathbf{k}$) then the electric field is contained in the plane

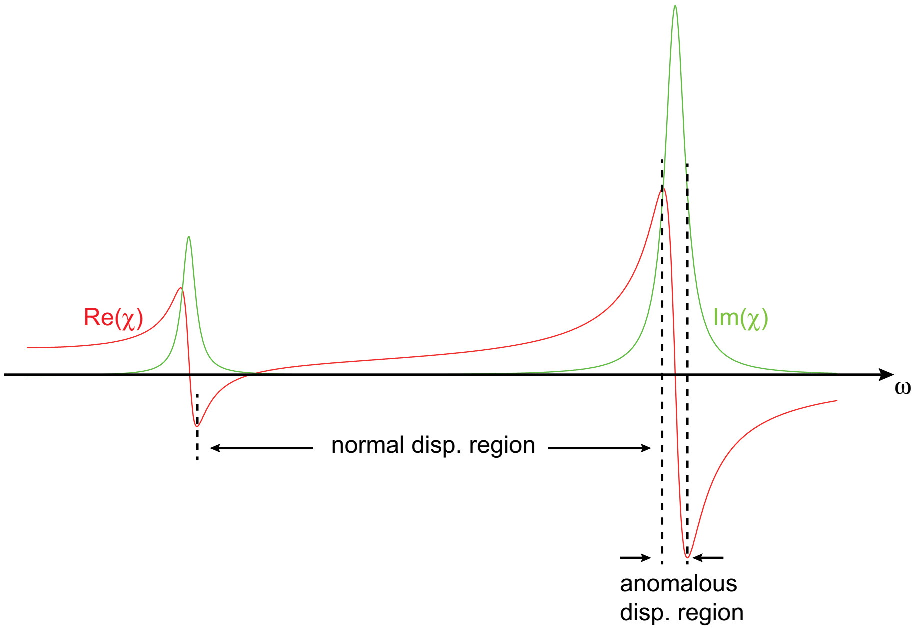



Figure 2: Real part and imaginary part of the susceptibility

$(xOy)$ and has two components $E_x$ and $E_y$, which oscillate at a frequency $\omega$, and the resulting electric field is a superposition of both contributions $\mathbf{E} = E_x\mathbf{e_x} + E_y\mathbf{e_y}$ where

$$E_x = A_x \cos(\omega t - kz) \tag{11a}$$
$$E_y = A_y \cos(\omega t - kz + \Delta) \tag{11b}$$

When $E_x$ and $E_y$ oscillate in phase ($\Delta = 0$), the resulting polarization is linear, and the orientation with respect to $(Ox)$ (Fig. 3). If we call $\theta$ the angle to the $x-$axis, then if $A_x = A_y$, $\theta = {}^{\pi}/4$ otherwise $\theta$ is defined by $\tan\theta = \left({}^{E_y}/E_x\right)$.

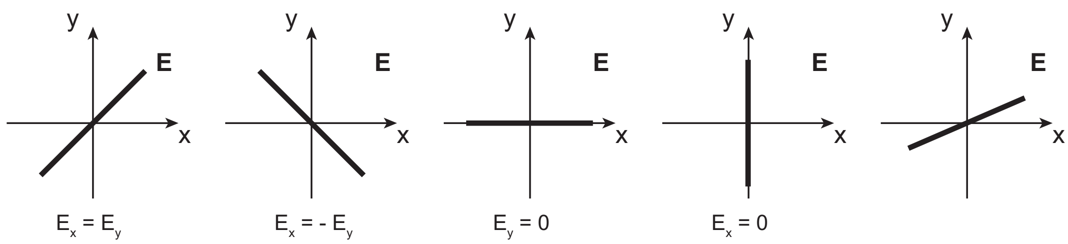



Figure 3: Various combination of $E_x$ and $E_y$ leading to linear polarization.

On the other hand, when $E_x$ and $E_y$ do not oscillate in phase ($\Delta \neq 0$), the projection onto the $(xOy)$ plane becomes an ellipse (Fig. 4).

As can be seen from Fig. 4, two types of circularly polarized beam can exist depending if $E_x$ is in advance with respect to $E_y$ or delayed. By definition, the beam is *left-hand* circularly polarized if the electric field of the beam coming towards the observer is describing a circle, rotating in the counterclockwise (Fig. 5), otherwise it is said to be *right-hand* circularly polarized.

### 0.1.3 Refraction and reflection with polarized light

#### Definition of the incident plane

For what follows it is important to properly define the *incident plane* since this also helps defining the electric field. If the electric field is orthogonal to the incident plane, the

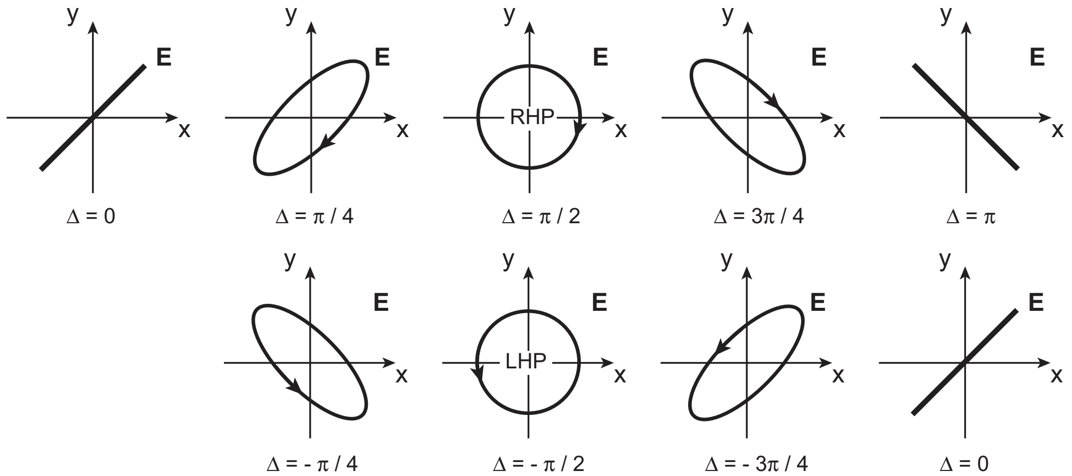



Figure 4: Superposition of $E_x$ and $E_y$, with $|E_x| = |E_y|$ for various dephasing $\Delta$ between $E_x$ and $E_y$.

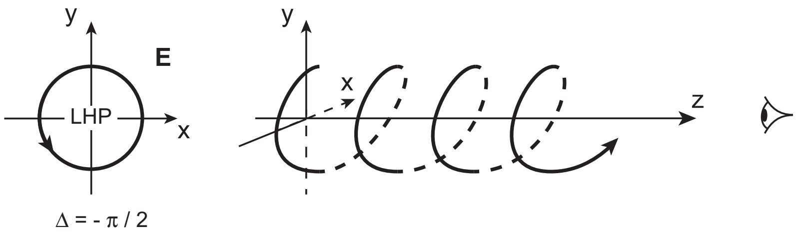



Figure 5: Left-hand circularly polarized light.

wave is defined as TE (*transverse electric* wave) whereas when the electric field belongs to that plane, the wave is defined as TM (*transverse magnetic* wave). As shown on Fig. the wavevector (direction of the propagation of the wave) and the normal of the surface, together with the incident point define the incident plane.

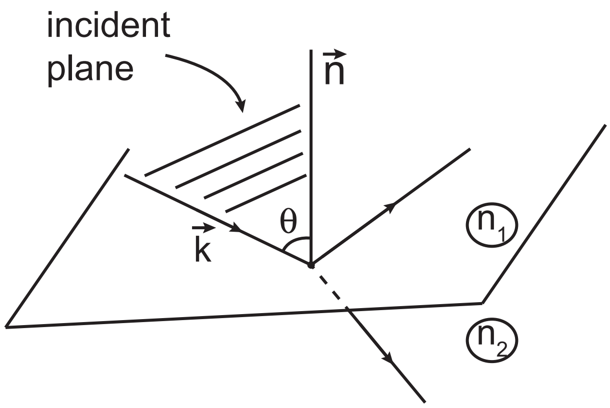



Figure 6: Definition of the incident plane.

We can define the reflection and transmission coefficient either in amplitude or in intensity. Defined in intensity, they are directly linked with the conservation of energy and $R+T=1$ where $R$ is the reflection coefficient and $T$ the transmission one. Usually small letter are used to express these coefficients in amplitude and $R=r^2$ and $T^2 = (n_1/n_2)\, t^2$.

Depending on the polarization of the wave the reflection coefficient are given by the

*Fresnel's relations*:

$$r_{\perp} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} E_{\perp} \tag{12a}$$

$$r_{\parallel} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} E_{\parallel} \tag{12b}$$

Two cases have to be considered depending on the relative magnitude of the refractive indices.

1. $\boxed{n_1 < n_2}$
At the Brewster angle ($i_B$), only the TE wave is reflected. Note also that below

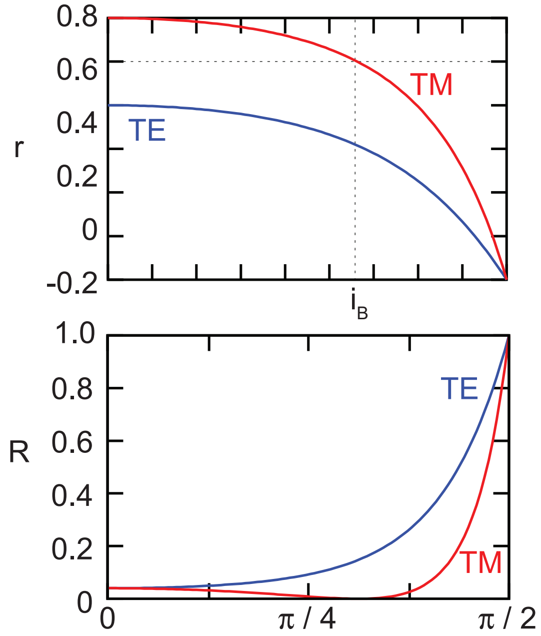



Figure 7: Reflections coefficient in amplitude (a) and in intensity (b)

the TM wave experience a change of phase-shift at the reflection depending if the angle is larger ($\pi-$phase-shift) or smaller (no phase-shift) than the Brewster angle. The TE wave always experience this $\pi-$phase shift at the reflection.

2. $\boxed{n_1 > n_2}$ In this case the situation is very different since there exist a maximum allowed angle: the *critical angle*. Above this angle, there is not refraction possible and the wave is totally reflected. this situation is actually used to coupled light into waveguide by so-called *evanescent coupling*.

### 0.1.4 Brewster Angle

When a randomly polarized arrives at a surface, if the incident angle is such that $\theta_i + \theta_r = (\pi/2)$ then the polarization state of the reflected beam can only be transverse electric. If we consider that the electric field create a polarization that plays the role of the emitter, it is obvious that the emitter for the TM wave cannot emit in the direction of the reflected beam. Only the TE wave is reflected. This is the *Brewster angle*.

## 0.2 Non isotropic medium

So far, we have only consider that the medium where the electric field propagated was isotropic, and therefore the dielectric susceptibility $\chi$ was the same for any direction. In

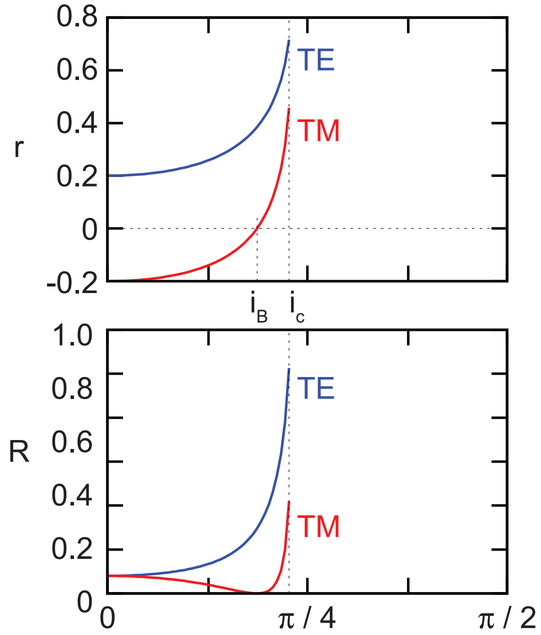



Figure 8: Reflections coefficient in amplitude (a) and in intensity (b)

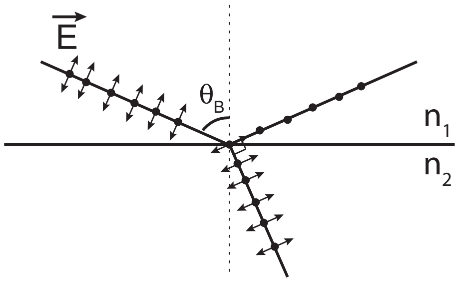



Figure 9: Definition of the Brewster angle

reality, this may be no true, and the displacement $\mathbf{D} = \epsilon_0(1+\chi)\mathbf{E} = \epsilon_0\epsilon(r)\mathbf{E}$ now involves a tensor for $\epsilon(r)$:

$$\mathbf{D} = \epsilon_0 \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{33} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \mathbf{E} \tag{13}$$

Mathematically, this matrix can be diagonalized, and this corresponds physically to find the principal axis (eigenvectors) of the medium. We can then write:

$$\mathbf{D} = \epsilon_0 \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \mathbf{E} = \epsilon_0 \hat{\epsilon_r} \mathbf{E} \tag{14}$$

where the tensor $\hat{\epsilon_r}$ consists in $\epsilon_i = n_i^2$, with $i = (x, y, z)$. From this tensor, it clear that there exists three situations, three types of material. For an isotropic material, $\forall i, n_i =$ cste. On the other hand, media in which all the three dielectric constants $\epsilon_i$ are difference are called *bi-axial*, whereas those in which only two of the three are different are called *uni-axial*.

In the general case, $\mathbf{k} \cdot \mathbf{E} \neq 0$ then the algebraic relation $\nabla \times \nabla \times \nabla E = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ applied to the Faraday law equation (Eq. (1a) leads to

$$\mathbf{k}^2\mathbf{E} - \mu_0\omega^2\hat{\epsilon}\ \mathbf{E} = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) \tag{15}$$

and considering the vectorial nature of both **k** and **E** we have Eq. (45):

$$\begin{aligned}\left(k_x^2+k_y^2+k_z^2\right)\left(E_x\mathbf{e_x}+E_y\mathbf{e_y}\ +\ E_z\mathbf{e_z}\right)-\mu_0\omega^2\epsilon_0\left(n_x^2E_x\mathbf{e_x}+n_y^2E_y\mathbf{e_y}+n_z^2E_z\mathbf{e_z}\right)&\\ =\left(k_x\mathbf{e_x}+k_y\mathbf{e_y}+k_z\mathbf{e_z}\right)\left(k_x\mathbf{e_x}+k_y\mathbf{e_y}+k_z\mathbf{e_z}\right)&\\ =k_x^2E_x\mathbf{e_x}+k_xk_yE_y\mathbf{e_y}+k_xk_zE_z\mathbf{e_z}&\\ +k_xk_yE_x\mathbf{e_x}+k_y^2E_y\mathbf{e_x}+k_xk_zE_z\mathbf{e_z}&\\ +k_xk_zE_X\mathbf{e_x}+k_yk_zE_z\mathbf{e_y}+k_z^2E_z\mathbf{e_z}&\end{aligned}\tag{16}$$

and on each axis:

$$/\mathbf{e_x}\ :\quad\left(k_y^2+k_z^2-\frac{n_x^2\omega^2}{c^2}\right)E_x=k_xk_yE_y+k_xk_zE_z\tag{17a}$$

$$/\mathbf{e_y}\ :\quad\left(k_x^2+k_z^2-\frac{n_y^2\omega^2}{c^2}\right)E_y=k_yk_xE_x+k_yk_zE_z\tag{17b}$$

$$/\mathbf{e_z}\ :\quad\left(k_x^2+k_y^2-\frac{n_z^2\omega^2}{c^2}\right)E_z=k_zk_xE_x+k_zk_yE_y\tag{17c}$$

These equations have a solution if and only if[1]

$$Det.=\begin{vmatrix}\left(k_y^2+k_z^2-\frac{n_x^2\omega^2}{c^2}\right) & -k_xky & -k_xk_z\\ -k_xk_y & \left(k_x^2+k_z^2-\frac{n_y^2\omega^2}{c^2}\right) & -k_yk_z\\ -k_xk_z & -k_yk_z & \left(k_x^2+k_y^2-\frac{n_z^2\omega^2}{c^2}\right)\end{vmatrix}=0\tag{18}$$

After calculation, we obtain:

$$\begin{aligned}Det.=\frac{-\omega^4}{c^4}+\frac{\omega^2}{c^2}\left(\frac{k_y^2+k_z^2}{n_x^2}+\frac{k_x^2+k_z^2}{n_y^2}+\frac{k_x^2+k_y^2}{n_z^2}\right)&\\ +\left(\frac{k_x^2}{n_y^2n_z^2}+\frac{k_y^2}{n_x^2n_z^2}+\frac{k_z^2}{n_x^2n_y^2}\right)\left(k_x^2+k_y^2+k_z^2\right)=0&\end{aligned}\tag{19}$$

### uni-axial crystal

In the general case, the three components $(n_x, n_y, n_z)$ are different, and we then refer these crystal to *bi-axial* crystal. But there is a simpler case, we , $n_x = n_y = n_o = n_\perp$ (the ordinary axis), and $n_z = n_e = n_\parallel$ (the extraordinary index). In this case, the Eq. (20) becomes

$$\begin{aligned}Det.=\frac{-\omega^4}{c^4}+\frac{\omega^2}{c^2}\left(\frac{k_y^2+k_z^2}{n_o^2}+\frac{k_x^2+k_z^2}{n_o^2}+\frac{k_x^2+k_y^2}{n_e^2}\right)&\\ +\left(\frac{k_x^2}{n_o^2n_e^2}+\frac{k_y^2}{n_o^2n_e^2}+\frac{k_z^2}{n_o^4}\right)\left(k_x^2+k_y^2+k_z^2\right)=0&\end{aligned}\tag{20}$$

[1] we remind that such determinant is calculated by:

$$\begin{vmatrix}a&b&c\\d&e&f\\g&h&i\end{vmatrix}=a\begin{vmatrix}e&f\\h&i\end{vmatrix}-d\begin{vmatrix}b&c\\h&i\end{vmatrix}+g\begin{vmatrix}b&c\\e&f\end{vmatrix}$$

which can be factorized as

$$\left(\frac{k_x^2+k_y^2+k_z^2}{n_o^2}-\frac{\omega^2}{c^2}\right)\left(\frac{k_x^2+k_y^2}{n_e^2}+\frac{k_z^2}{n_o^2}-\frac{\omega^2}{c^2}\right)=0 \tag{21}$$

The first term of this equation corresponds to a sphere whereas the second represents an ellipsoid. The difference between the refractive indices of the ordinary and the extraordinary beam is known as the *birefringence* $\Delta n = n_e - n_o$. Depending of its sign, the crystal is said to be *positive* or *negative* (Fig. 10).

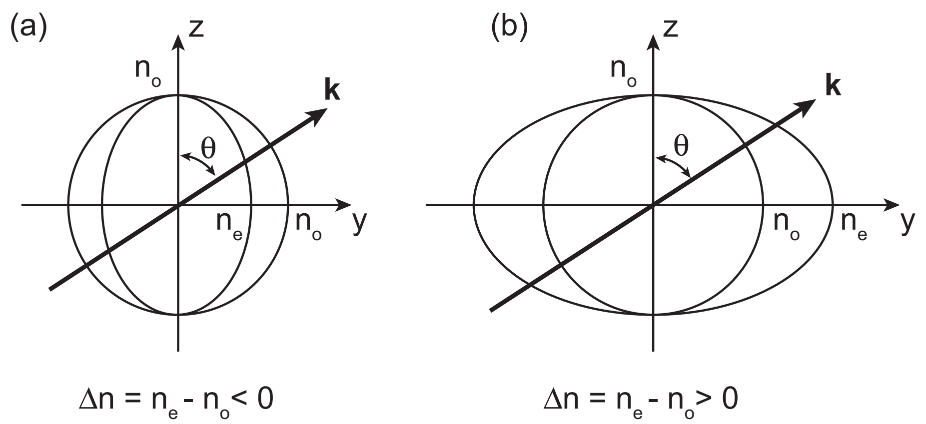



Figure 10: Representation of the *index ellipsoid* cut in the plane $(O_x)$ for (a) negative and (b) positive crystal. **k** coincides with the optical axis.

## 0.3 Jones' formalism - representation of the polarization

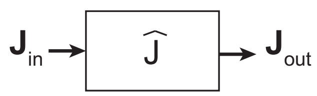
Since the electric field is transverse, it can simply written as

$$\mathbf{E}(z,t)=\begin{pmatrix} A_x\cos(\omega t-kz+\delta_x)\\ A_y\cos(\omega t-kz+\delta_y)\\ 0\end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} A_x e^{i\delta_x}\\ A_y e^{i\delta_y}\\ 0\end{pmatrix} \mathrm{e}^{\mathrm{i}(\omega \mathrm{t}-\mathrm{kz})}\right] \tag{22}$$

In order to see the evolution of the polarization state as the wave is propagating, Jones established in 1941 a useful formalism, in which the polarization field is simply describe as

$$\mathbf{J}=\begin{pmatrix} J_x\\ J_y\end{pmatrix}=\frac{1}{\sqrt{A_x^2+A_y^2}}\begin{pmatrix} A_x e^{i\delta_x}\\ A_y e^{i\delta_y}\end{pmatrix} \tag{23}$$

The norm of the Jones' matrix is $JJ^*=1$. Using this formalism, the polarization of the incident beam can be described and follow as the beams goes through the various element that may affect the polarization. Each element will be represented by a $2\times 2$ matrix:




$$\text{with} \quad \left\{\begin{array}{lcl} J_{\mathrm{out}}^x & = & A\ J_{\mathrm{in}}^x + B\ J_{\mathrm{in}}^y \\ J_{\mathrm{out}}^y & = & C\ J_{\mathrm{in}}^x + D\ J_{\mathrm{in}}^y \end{array}\right.$$

### Particular states of polarization

The different states of polarization can be expressed as

linear polarization

$$\mathbf{E} = E_x \mathbf{e_x} \Longrightarrow \mathbf{J} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{E} = E_y \mathbf{e_y} \Longrightarrow \mathbf{J} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\textit{linear at } 45^o \quad \mathbf{E} = E_0\ \mathbf{e_x} + E_0\ \mathbf{e_y} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\textit{linear at } -45^o \quad \mathbf{E} = E_0\ \mathbf{e_x} - E_0\ \mathbf{e_y} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

circular polarization

$$|E_x| = |E_y| \text{ and } (\delta_{\mathrm{x}} - \delta_{\mathrm{y}}) = \pm\frac{\pi}{2} \Longrightarrow \mathbf{J} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

## 0.3.1 basics polarizing element & Jones calculus

### Polarizer

A polarizer is an element which produces linear polarization from any arbitrary polarization states. Somehow, this is a simple projection of the polarization state on the axis of the polarizer. If the polarizer is along $\mathbf{e}_x$ (resp. $\mathbf{e}_y$) then the Jones' matrix is

$$\hat{\mathbf{J}}_{\mathbf{P}\|\mathbf{e_x}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{\mathbf{J}}_{\mathbf{P}\|\mathbf{e_y}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

If the polarizer is at $\pm 45^o$ then:

$$\hat{\mathbf{J}}_{\mathbf{P}[+45^o]} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \hat{\mathbf{J}}_{\mathbf{P}[-45^o]} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{26}$$

### Use of a polarizer

1. Let's assume that we have a beam that is linearly polarized along the $\mathbf{e}_x$ axis and the axis of the polarizer is set at $45^o$:

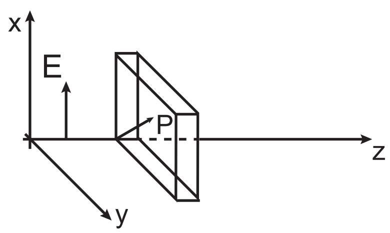



Figure 11: action of a polarizer set at $45^o$

The resulting polarization state can be readily calculated by

$$J_{\text{out}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{27}$$

The resulting intensity is $(1/2)\,E_0^2$: half of the intensity is lost in the process.

2. Another interesting case is when we use two polarizers orthogonal to each other, in a so-called *crossed* configuration. The resulting transfer matrix is

$$\hat{J}_{\text{crossed}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{28}$$

   No light can pass through!

3. In the general case where a polarizer is set at an angle $\alpha$ then the resulting polarization state can be derived using the rotational transformation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

   then the final polarization state can be calculated as

$$J_{\text{out}} = R(-\alpha)\ \hat{\mathbf{J}}_{P\parallel \mathbf{e}'_{\mathbf{x}}}\ R(\alpha) J_{\text{in}} \tag{29}$$

**Waveplate - retarder**

Let us now assume that we insert on the path of the beam a plate made of a birefringent material. The plate is oriented such that it optic axis is along $\mathbf{e_x}$ (Fig. 12). Within such a configuration the refractive index along the $x-$axis is $\mathrm{n}_{\parallel}$ and the one along the $y-$axis is $\mathrm{n}_{\perp}$.

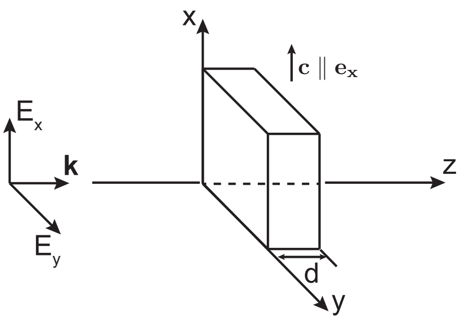



Figure 12: Schematic of the setup using a waveplate. The optic axis $\mathbf{c}$ is colinear with $\mathbf{e_x}$. The thickness of the plate is $d$.

The wave incident to that plate is defined as $E = E_0 \exp i\,(\omega_0 t - kz + \varphi_0)$. After the propagation in the birefringent material the transverse component of the electric fields have experienced a different phase shift so that

$$E_x(z = d) = E_{0x} \exp i\,(\omega t - k_0 n_{\parallel} d + \varphi_{0x}) \tag{30a}$$
$$E_y(z = d) = E_{0y} \exp i\,(\omega t - k_0 n_{\perp} d + \varphi_{0y}) \tag{30b}$$

At the output of the waveplate the phase different between the $x-$component and the $y-$component of the electric field is

$$\Delta\varphi = \Delta\varphi_0 + (k_0 n_{\parallel} - k_0 n_{\perp})d \tag{31}$$

where $\Delta\varphi_0 = \varphi_{0y} - \varphi_{0x}$ is the phase different between the components $E_x$ and $E_y$ at the input of the waveplate. Note that since in a birefringent material $n_{\parallel} \neq n_{\perp}$ then the phase shift induced during the propagation is not the same on both axis. For this reason such optical element is sometimes called a *retarder*.

### Jones matrix for a waveplate

Considering that the induced phase for each transverse component of the electric field is now determined (eq. (30a)) we can readily translate this into Jones' formalism:

$$\mathbf{J}_{\text{out}} = \begin{pmatrix} E_x e^{-ik_0 n_\parallel d} \\ E_y e^{-ik_0 n_\perp d} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-ik_0 n_\parallel d} & 0 \\ 0 & e^{-ik_0 n_\perp d} \end{pmatrix}}_{\hat{J}} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{32}$$

We can also write the Jones' matrix in a more symmetric way as

$$\hat{J} = e^{i\frac{\Delta\varphi'}{2}} \begin{pmatrix} e^{-i\frac{\Delta\varphi'}{2}} & 0 \\ 0 & e^{i\frac{\Delta\varphi'}{2}} \end{pmatrix} \tag{33}$$

where $\Delta\varphi'$ is phase shift induced

$$\Delta\varphi' = k_0(n_\parallel - n_\perp)d = k_0 d\delta n \tag{34}$$

### half-wave plate

Let us now assume that the retardation is $\pi$. For an electric field polarised at $45^o$ (*i.e.* $|E_x| = |E_y|$ and $\Delta\varphi_0 = 0$) then

$$\Delta\varphi' = d\frac{2\pi}{\lambda}\delta n = \pi + 2m\pi \qquad \text{with } m \in \mathbb{Z}^* \tag{35}$$

$$\Rightarrow \; d\cdot\delta n = \frac{\lambda}{2}(1+2m) \tag{36}$$

Such a plate is called a *half-wave plate*. $m$ is called the order of the plate. In practise $\delta n$ is determined by the material but the thickness of the plate can be chosen. For $m = 0$ we talk about *zero-order* waveplate. They are usually more expansive and very fragile. According to the eq. (33) the Jones' matrix for a half-wave plate is

$$\hat{J}_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{37}$$

**Use of a half-wave plate.** Note that we calculate the influence of the waveplate in a very particular case: the input beam is polarized along the optic axis of the waveplate. In general this is not the case. Let us assume that the axis of the plate is rotated by an angle $\alpha$ with respect to the linear polarization of the input beam (Fig.). We should not use the eq. (29). In other word we need first to align the electric field with the optic axis of the waveplate, then apply the waveplate and return in the original frame of work:

$$J'_{in} = R(\alpha)J_{in} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ 0 \end{pmatrix} = \begin{pmatrix} E\cos\alpha \\ -E\sin\alpha \end{pmatrix} \tag{38}$$

and after the waveplate

$$J'_{out} = \begin{pmatrix} E\cos\alpha \\ E\sin\alpha \end{pmatrix} \tag{39}$$

and finally

$$J_{out} = R(-\alpha)J'_{out} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E\cos\alpha \\ E\sin\alpha \end{pmatrix} = E\begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} \tag{40}$$

The beam is still linearly polarized but the orientation of its polarization has been rotated by $2\alpha$!

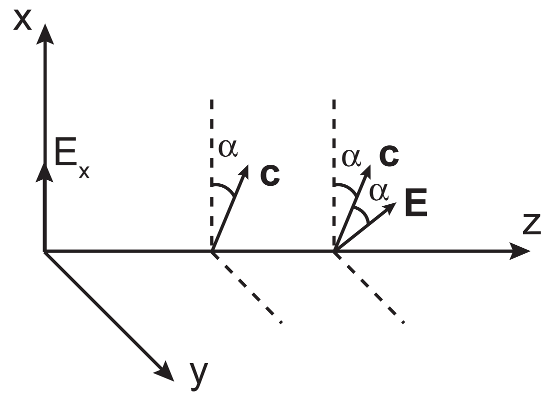



Figure 13: Effect of a half-wave plate, for which the optic axis **c** makes an angle $\alpha$ with the direction of the electric field incident on the waveplate.

**quarter-wave plate**

Another important optical element is the quarter-wave plate. By contrast with the half-wave plate the induced phase shift is now ($\pi/2$). A wave linearly polarized at $45^o$ (*i.e.* $|E_x| = |E_y|$ and $\Delta\varphi_0 = 0$) will transform into a circularly polarized wave according Fig. 4. Using the eq. (34) we can readily obtain

$$d \cdot \delta n = \frac{\lambda}{4}(1+2m) \qquad \text{with } m \in \mathbb{Z}^* \tag{41}$$

and the Jones' matrix is

$$\hat{J}_{\lambda/4} = e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{42}$$

As for the half-wave plate the orientation of the quarter-wave plate is important. If the linearly polarized incident beam is such that its electric field is along (or orthogonal to) the optic axis of the quarter-wave plate then the plate has no influence. By contrast it the input field is linearly polarized at $45^o$ then the situation is very different:

$$\mathbf{J}_{out} = e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{43}$$

The beam is indeed circularly polarized.

### 0.3.2 Representation of polarization state: the Poincaré sphere

Fig. 4 represents all the possible polarization state (besides for unpolarized light). One way to determine precisely the state of polarization of a beam is to use the four *Stokes parameters*, introduced by G.G. Stokes in 1852. The main advantage of these parameters is that they can be fully determine through intensity measurement, and can therefore be straightforwardly been measured. With the definition of the field (Eq. (11)), the Stokes parameters are

$$s_0 = |\mathbf{E}.\mathbf{e_x}|^2 + |\mathbf{E}.\mathbf{e_y}|^2 = A_x^2 + A_y^2 \tag{44a}$$
$$s_1 = |\mathbf{E}.\mathbf{e_x}|^2 - |\mathbf{E}.\mathbf{e_x}|^2 = A_x^2 - A_y^2 \tag{44b}$$
$$s_2 = 2\mathrm{Re}\left[(\mathbf{E}.\mathbf{e_x})^*(\mathbf{E}.\mathbf{e_y})\right] = 2\mathrm{A_x A_y}\cos\Delta \tag{44c}$$
$$s_3 = 2\mathrm{Im}\left[(\mathbf{E}.\mathbf{e_x})^*(\mathbf{E}.\mathbf{e_y})\right] = 2\mathrm{A_x A_y}\sin\Delta \tag{44d}$$

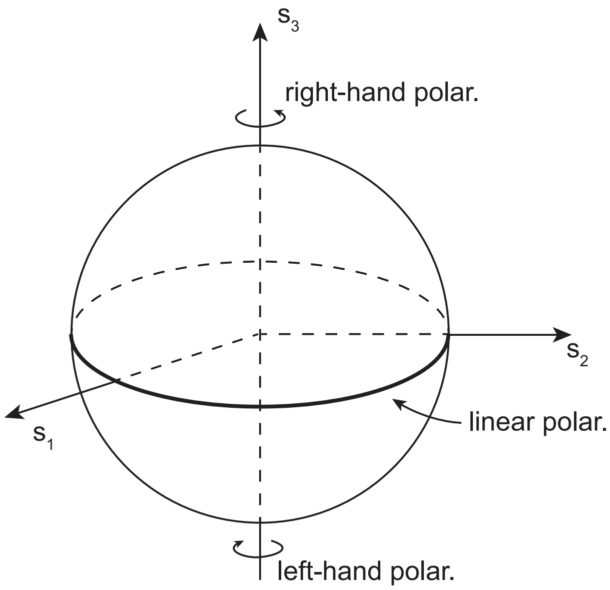



Figure 14: Poincaré sphere representing the possible states of polarization.

Note that $s_0^2 = s_1^2 + s_2^3 + s_3^2$ is the total intensity of the wave, $s_1$ gives the preponderance of $x-$linear polarization over the $y-$polarization, and $s_2$ and $s_3$ give the phase information. Finally, one way to visualize the Stokes parameter is to use the *Poincaré sphere* introduce by Henri Poincaré in 1892 (Fig. 14).

# .1 index ellipsoid - *alternative calculation*

Since in the general case, $\mathbf{k}.\mathbf{E} \neq 0$ then the algebraic relation $\nabla\times\nabla\times\nabla E = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E}$ applied to the Faraday law equation (Eq. (1a) leads to

$$\mathbf{k}^2\mathbf{E} - \mu_0\omega^2\hat{\epsilon}\ \mathbf{E} = \mathbf{k}(\mathbf{k}.\mathbf{E}) \tag{45}$$

Therefore on the axis $(O_x)$:

$$\begin{aligned} /(O_x) \qquad & \mathbf{k}^2 E_x - \mu_0\omega^2 D_x = k_x(\mathbf{k}.\mathbf{E}) \\ \Leftrightarrow \quad & \mathbf{k}^2\frac{D_x}{\epsilon_1} - \mu_0\omega^2 D_x = k_x(\mathbf{k}.\mathbf{E}) \\ \Leftrightarrow \quad & \frac{D_x}{\epsilon_1}\left(\mathbf{k}^2 - k_1^2\right) = k_x(\mathbf{k}.\mathbf{E}) \end{aligned} \tag{46}$$

Since the medium is a dielectric (no charge), the Maxwell-Gauss equation $\nabla.\mathbf{D} = 0$ leads to $\sum_{i=x,y,z} k_i D_i = 0$:

$$\epsilon_1\frac{k_x^2}{k^2 - k_1^2} + \epsilon_2\frac{k_y^2}{k^2 - k_2^2} + \epsilon_3\frac{k_z^2}{k^2 - k_3^2} = 0 \tag{47}$$

which can then be rewritten as

$$n_1^2\frac{n_x^2}{n^2 - n_1^2} + n_2^2\frac{n_y^2}{n^2 - n_2^2} + n_3^2\frac{n_z^2}{n^2 - n_3^2} = 0 \tag{48}$$

This is an ellipsoid and it characterises the anisotropic medium. Such surface is actually hard to visualize, especially in the general case, where the three semi-axis of the *index ellipsoid* $n_1, n_2, n_3$ are different. One way to represent this ellipsoid is actually to cut it by the $(O_{yz})$ plane $(n_x = 0)$. Note that to safely do $n_x = 0$ we need to consider both cases: (i) $n^2 = n_1^2$ and (ii) $n^2 \neq n_1^2$.

$\boxed{n^2 - n_1^2 \neq 0}$

Eq. (48) can simply be written as:

$$\begin{aligned} & n_2^2 n_y^2\left(n^2 - n_3^2\right) + n_3^2 n_z^2\left(n^2 - n_2^2\right) = 0 \\ \Leftrightarrow \quad & n_2^2 n_y^2 n^2 + n_3^2 n_z^2 n^2 = n_2^2 n_3^2 n_y^2 + n_3^2 n_2^3 n_z^2 = \left(n_y^2 + n_z^2\right) n_2^2 n_3^2 \\ \Leftrightarrow \quad & \frac{n^2 n_y^2}{n_3^2} + \frac{n_z^2 n^2}{n_2^2} = n_y^2 + n_z^2 \end{aligned} \tag{49}$$

and since $n^2 = n_x^2 + n_y^2 + n_z^2$ we obtain

$$\frac{n_y^2}{n_3^2} + \frac{n_z^2}{n_2^2} = 1 \tag{50}$$

This is the equation of an ellipse!

$\boxed{n^2 - n_1^2 = 0}$

Similar calculation leads this time to the equation of a circle

$$n_y^2 + n_z^2 = n_1^2 \tag{51}$$