Chapter 2

Gaussian Optics

The goal of this part is to introduce the basics of Gaussian optics. We will focus first on the Matrices formalism as this is what is used in the study of propagation of beam as well as in cavity design (next chapter).

2.1 Ray matrices

2.1.1 Ray matrices

The formalism of ray matrices is based on the paraxial approximation: the *optical rays remain confined around the optical axis*. In this approximation, and since the transverse dimension are always smaller than the longitudinal distances, the angle θ between this optical axis and the direction of the ray is

$$\sin\theta \simeq \tan\theta \simeq \theta \tag{2.1}$$

In this approximation, any optical system can be determined by an ABCD transfer matrix, which gives the relationship between the beam (distance to the axis r_1 and the incident angle θ_1) at the input plane (Π_1) and the beam at the output (Π_2).

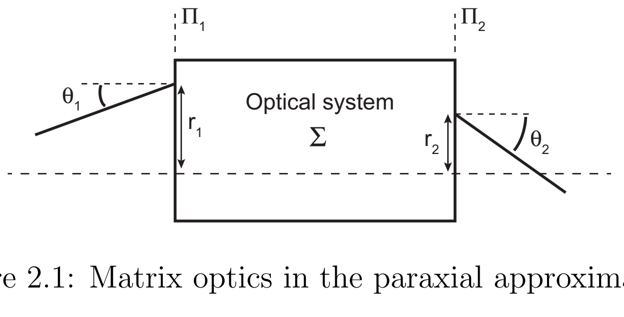

Figure 2.1: Matrix optics in the paraxial approximation

Note that in the paraxial approximation (small angles) the incident angle is $\theta = dr/dz = r'$ (fig. 2.1). At the output of the optical system (Π_2), the beam is characterised by (r_2, r'_2) with

$$\begin{pmatrix} r_2 \\ r'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} r_1 \\ r'_1 \end{pmatrix} \tag{2.2}$$

ABCD matrix is a very powerful tool and it encloses many properties of the optical system (position of the principal planes, of the nodal points, focal distances ... etc). The goal here is not to list all these properties. Actually we are only interesting by applying this tool to laser cavities. We should however point out one important property: the determinant of the transfer matrix between Π_1 and Π_2 is given by the ratio of the refractive indices

$$\det(ABCD) = \frac{n_1}{n_2} \tag{2.3}$$

where n_1 (resp. n_2) is the refractive index of the input (resp. output) materials.

Let us now focus on the transfer matrix for the basic elements that will be needed in the study of laser cavities:

free-space

In the simple case of free path (the beam remains in the same medium), simple geometry gives (fig. 2.2):

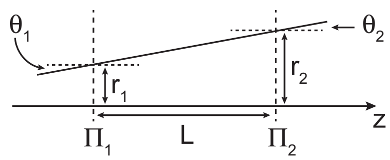

Figure 2.2: Ray optics in the case of free path

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

Note that the situation is different if the light propagates in a material with a refractive index n but is enclosed in air (Fig. 2.3).
In this case the transfer matrix[1] is given by

$$M = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix} \tag{2.5}$$

[1] see exercises.

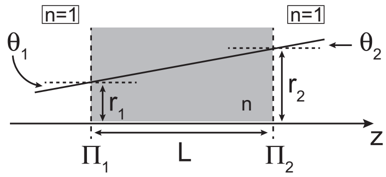

Figure 2.3: Ray optics for free-space but inside a material. The material has a refractive index n and is enclosed in air ($n = 1$).

This contrast with the matrix (eq. (2.4) as the object appears closer that it is by a factor n. The distance L/n is sometimes referred as the *effective distance*. This differs from the *optical distance* which is nL.

Thin lenses

Fig. 2.4 presents the construction of a beam passing through a thin lens. From

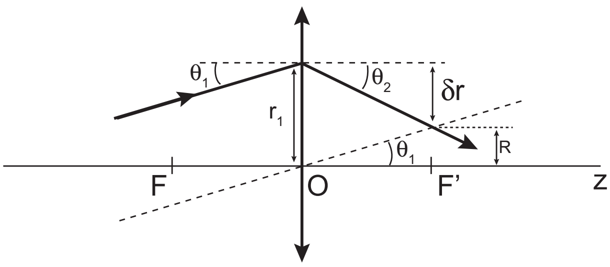

Figure 2.4: Transfer matrix for a thin lens.

this figure, we can write

$$\theta_2 = \frac{-\delta_r}{f} \tag{2.6a}$$
$$\delta_r = r_1 - R \tag{2.6b}$$
$$R = F \tan \theta_1 \sim F \times \theta_1 \tag{2.6c}$$

By expressing δ_r

$$\delta_r = r_1 - f\theta_1 \rightarrow \theta_2 = \theta_1 - (r_1/f) \tag{2.7}$$

and taking into account that $r_1 = r_2$ we can obtain:

$$M = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \tag{2.8}$$

Of course we can also use this transfer matrix in the case of an optical mirror. In that case the focal length is simply given by $f = (R/2)$, where R is the radius of curvature of the mirror.

Transfer at an interface

From Snell-Descartes' law of refraction

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{2.9}$$

in the small-angle approximation, we can simply write

$$n_1\ \theta_1 = n_2\ \theta_2 \tag{2.10}$$

and then the transfer matrix at an interface is given by:

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} r_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ \theta_1 \end{pmatrix} \Rightarrow \begin{cases} r_2 = r_1 \\ \theta_2 = \dfrac{n_1}{n_2}\ \theta_1 \end{cases} \tag{2.11}$$

at an interface we have

$$M = \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix} \tag{2.12}$$

"duct"

A duct is an element for which the refractive index depends transversely according to a quadratic law:

$$n(x) = n_0 - 1/2 n_2 x^2 \tag{2.13}$$

with $\gamma = n_2/n_0$.

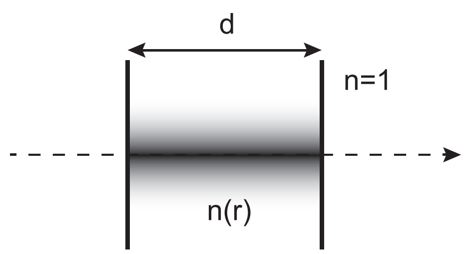

Figure 2.5: Gradient-index material - "duct". Its length is d.

For such component we have the transfer matrix:

$$M = \begin{pmatrix} \cos\left(d\sqrt{\frac{n_2}{n_0}}\right) & \frac{1}{\sqrt{n_0 n_2}}\sin\left(d\sqrt{\frac{n_2}{n_0}}\right) \\ -\sqrt{n_0 n_2}\sin\left(d\sqrt{\frac{n_2}{n_0}}\right) & \cos\left(d\sqrt{\frac{n_2}{n_0}}\right) \end{pmatrix} \tag{2.14}$$

Use of the ABCD matrix

Let consider an optical system, and study at B its influence on an optical beam starting from A (fig. 2.6). The transfer matrix from A to B is:

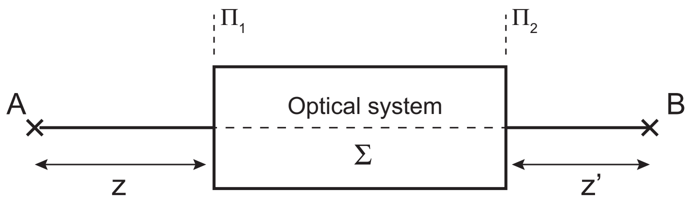

Figure 2.6: Influence of an optical system on a beam from A to B.

$$\begin{pmatrix} r_I \\ r'_I \end{pmatrix} = M_{A\to I} \times \begin{pmatrix} r_A \\ r'_A \end{pmatrix} \tag{2.15}$$

$$\begin{pmatrix} r_O \\ r'_O \end{pmatrix} = M_z \times \begin{pmatrix} r_I \\ r'_I \end{pmatrix} = M_\Sigma \times M_{A\to I} \times \begin{pmatrix} r_A \\ r'_A \end{pmatrix} \tag{2.16}$$

$$\begin{pmatrix} r_O \\ r'_O \end{pmatrix} = M_z \times \begin{pmatrix} r_I \\ r'_I \end{pmatrix} = M_\Sigma \times M_{A\to I} \times \begin{pmatrix} r_A \\ r'_A \end{pmatrix} \tag{2.17}$$

$$\begin{pmatrix} r_B \\ r'_B \end{pmatrix} = M_{O\to B} \times \begin{pmatrix} r_O \\ r'_O \end{pmatrix} = M_{O\to B} \times M_\Sigma \times M_{A\to I} \times \begin{pmatrix} r_A \\ r'_A \end{pmatrix} \tag{2.18}$$

Figure 2.7: $M_\Sigma = M_N \times M_{N-1} \cdots \times M_1$

Note that the product of matrices is written backwards compare to the order of the element met by the beam.

2.2 Maxwell's equations

Maxwell's equation are the basis for a proper description of an electromagnetic wave. They link the different quantities:

$\mathbf{E}$:	electric field	$[E] = V \cdot m^{-1}$
$\mathbf{H}$:	magnetic field	$[H] = A \cdot m^{-1}$
$\mathbf{D}$:	electric displacement	$[D] = A \cdot s \cdot m^{-2}$
$\mathbf{B}$:	magnetic induction	$[B] = V \cdot s \cdot m^{-2}$
$\mathbf{j}$:	electric current density	$[J] = A \cdot m^{-2}$
ρ :	electric charge density	$[\rho] = A \cdot s \cdot m^{-3}$

Note that all quantities are function of the spatial coordinate (x, y, z) as well as the time t. Using the nabla operator

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Maxwell equations are formulated as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.19a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.19b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2.19c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.19d}$$

The meaning of each equation is quite clear:

(2.19a) the Faraday law: The variation of the magnetic induction **B** creates vortexes of the electric field.

(2.19b) the Ampere law: voices of the magnetic field **H** are either caused by an electric current with a density **j** or by a temporal variation of the electric displacement **D**. $\nabla_t \mathbf{D}$ is the electric displacement current.

(2.19c) the Gauss law: the sources of the electric displacement **D** are the electric charges with density ρ.

(2.19d) the magnetic field (induction) is solenoidal, *i.e., there exists no* magnetic charges.

To these equations are associated the so-called *constitutive relations* for the material:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} \tag{2.20a}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu_0 \mu_r \mathbf{H} \tag{2.20b}$$

where ϵ_r (resp. μ_r) is called the relative *permitivity* (resp. *permeability*). Note that in the most general case ϵ_r is a tensor. For non magnetic materials $\mu_r = 1$ and can be dropped. The vacuum permitivity ϵ_0 and vacuum permeability μ_0 are given by

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F \cdot m^{-1}} \simeq \frac{1}{36\pi \times 10^{-6}} \qquad \left(\mathrm{A \cdot s}/\mathrm{V \cdot m}\right) \tag{2.21}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H \cdot m^{-1}} \simeq 1.256 \times 10^{-6} \qquad \left(\mathrm{V \cdot s}/\mathrm{A \cdot m}\right) \tag{2.22}$$

2.2.1 propagation in vacuum

In vacuum ($\epsilon_r = 1$) and for non-magnetic materials the equations become

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2.23a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.23b}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{2.23c}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.23d}$$

Using the algebraic equation $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.A) - \nabla^2 \mathbf{A}$ applied to eq. (2.23a), we derive the equation for the propagation of the electric field[2]:

$$\left[\nabla^2 - \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \mathbf{E} = \underbrace{\nabla \left(\rho/\epsilon_0\right) + \mu_0 \frac{\partial \mathbf{J}}{\partial t}}_{\text{source terms}} \tag{2.24}$$

[2]To find the equation of propagation of the field from Maxwell's equation we need to use the algebraic identity:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

applied to eq. (2.23a):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \left(\rho/\epsilon_0\right) - \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \nabla \left(\rho/\epsilon_0\right) - \mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H})$$

and use the eq. (2.23b).

Finally, in an isotropic material without any charge nor loss (no density of current) the equation of propagation becomes:

$$\nabla^2 E - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0 \tag{2.25}$$

The speed of the wave (phase-velocity) is given by

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \tag{2.26}$$

2.2.2 linear propagation in a material

For an isotropic material, we can simply replace ϵ_0 by $(1+\chi_e)\epsilon_0 = \epsilon_r \epsilon_0$ and obtain the equation of propagation

$$\nabla^2 E - \epsilon_r \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \nabla^2 E - \frac{\epsilon_r}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \tag{2.27}$$

The phase velocity is in that case:

$$v_\varphi = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{1+\chi_e}} = \frac{c}{n} \tag{2.28}$$

where n is the refractive index of the material.

2.3 Solutions of the propagation equation

2.3.1 Helmholtz equation

As we saw previously, the propagation of an electromagnetic wave in vacuum is described by the equation

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \tag{2.29}$$

Note that in the current case the phase velocity is c and it is constant. There is no dispersion here. Since this is a spatio-temporal equation (partial differential equation), it is legitimate to try solving it by separating the variable:

$$\mathbf{E}(\mathbf{r}, t) = A(\mathbf{r}) f(t) \tag{2.30}$$

where $A(\mathbf{r})$ only depends on the spatial coordinate $\mathbf{r}$ and $f(t)$ is a function of the time. Inserting this ansatz into eq. (2.29) yields

$$\left[\nabla^2 A f(t) - \frac{1}{c^2} A(\mathbf{r}) \frac{\partial^2 f(t)}{\partial t^2}\right] \cdot A(\mathbf{r}) f(t) = 0$$

$$\Rightarrow \qquad \frac{\boldsymbol{\nabla}^{\mathbf{2}} A(\mathbf{r})}{A(\mathbf{r})^2} = \frac{1}{c^2} \frac{1}{f(t)} \frac{\partial^2 f(t)}{\partial t^2} \tag{2.31}$$

Since each side of the eq. (2.31) only depends on one variable, they must be both equal to the same constant value. For convenience we take this value equal to $-k^2$. Let us first look at the time-dependent function $f(t)$. It must be solution of the differential equation

$$\frac{1}{c^2}\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} = -k^2 \quad \Rightarrow \quad \frac{d^2 f}{dt^2} + k^2 c^2 f = 0 \tag{2.32}$$

This has a periodic solution of the form $\exp(i\omega t)$ where $\omega^2 = k^2 c^2$ is the angular frequency of the function $f(t)$. The LHS of the eq. (2.31) on the other hand yields the so-called *Helmholtz equation*:

$$\boxed{\left[\nabla^2 + k^2\right] A(\mathbf{r}) = 0} \tag{2.33}$$

The relation between k and ω is called the dispersion relation. It is interesting to note that the Helmholtz equation takes care of the dispersion and is valid even with the phase velocity is not constant.

2.3.2 The plane wave solution

The easiest solution of the Helmholtz equation is the plane wave

$$A(\mathbf{r}) = A_0 e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.34}$$

where A_0 is the complex envelope and is a constant value and $\mathbf{k} = (k_x, k_y, k_z)$ is the wave-vector indicating the direction of propagation of the wave. The phase-front of the that wave is simply given by $\varphi(\mathbf{r}$ =constant, corresponding to $\mathbf{k}\cdot\mathbf{r}$ =constant. These are planes. The norm of $\mathbf{k}$ is the wave-number $|\mathbf{k}|$ such that $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 = (\omega/c)^2$.

Since we took the convention that $f(t) \propto \exp(-i\omega t)$ the wave described by eq. (2.34) is traveling in the forward direction. Along $z-$direction it is :

$$E = \frac{1}{2}\left(U + U^*\right) = |A_0| \cos\left[kz - \omega t + \arg(A_0)\right] \tag{2.35}$$

Of course such wave is an idealized solution since the intensity $I(r) \propto |A_0|^2$ is constant everywhere! Note that we can write the argument of the cosine function as $(z/c - t)\,\omega$, which clearly shows that we have a wave traveling in the $+z$ direction (forward direction) at a velocity c. The Helmholtz equation has another plane wave solution $\propto \exp\left[-i\left(\omega t + kz\right)\right]$ corresponding to a wave traveling along z but in the backward direction.

In a material with a refractive index n, the phase-velocity is $v_\varphi = c/n$ and the wavelength is scaled by n such that $\lambda_{\text{mat.}} = \lambda_0/n$. In the optics domain $\lambda \in [400, 800]$ nm (visible). This yields the optical frequency $\nu \in [375, 750]$ THz or the angular frequency $\omega = 2\pi\nu \in [2.3 \times 10^{15}, 4.7 \times 10^{15}]$ rad/s.

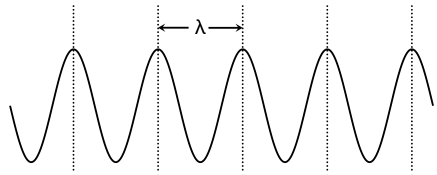

Figure 2.8: representation of a plane wave

2.3.3 Paraxial approximation / Gauss approximation in ray optics

Obviously the laser beam cannot be described as a plane wave although a collimated laser beam propagates straight. Indeed it also diverges due to diffraction during the propagation. Although it behaves along the direction of propagation ($z-$axis) nearly as a plane wave, instead of having a constant amplitude like the plane wave its amplitude depends on (x, y) and is spatially localized around the $z-$axis. Mathematically such a wave is described as[3]

$$\mathbf{E}(x,y,z,t) = \frac{1}{2}\, A(x,y,z)e^{i\omega t}\mathbf{n} + c.c. \tag{2.36}$$

where

$$A(x,y,z) = F(x,y,z)e^{-ikz} \tag{2.37}$$

In this form $F(x, y, z)$ is the complex amplitude. Note that in this form we can assume that the phase term evolve at a different rate as the complex amplitude. In particular the paraxial wave must fulfill the following condition: after a distance $\Delta z = \lambda$ the change $\Delta F(x, y, z)$ is much smaller than $F(x, y, z)$ itself. Since the evolution of $F(x, y, z)$ along the propagation is

$$\Delta F = \left(\frac{\partial F}{\partial z}\right)\Delta z = \left(\frac{\partial F}{\partial z}\right)\lambda \tag{2.38}$$

then $\Delta A \ll A \Leftrightarrow \left(\partial A/\partial z\right) \ll \left(A/\lambda\right) = \left(Ak/2\pi\right)$. Such approximation is called the *paraxial approximation* or the *Gauss approximation.* Obviously within this approximation, the first derivative should also vary slowly within the distance λ so that in conclusion we can write

$$\left|\frac{\partial F}{\partial z}\right| \ll k\,|F| \tag{2.39a}$$

$$\left|\frac{\partial^2 F}{\partial z^2}\right| \ll k^2\,|F| \tag{2.39b}$$

[3]we assume here that the beam is linearly polarised field and we simply use the scalar form

In these conditions, inserting eq. (2.37) into the eq. (2.33) yields [4]

$$\triangle_\perp F - 2ik\frac{\partial F}{\partial z} = 0 \tag{2.41}$$

which is called the *parabolic equation*. This equation is the fundamental equation to study the Gaussian optics. Note that we can write this equation in the following form:

$$\frac{\partial F}{\partial z} = -\frac{i}{2k}\triangle_\perp F \tag{2.42}$$

In this form, it is clear that the transverse Laplacian $\triangle_\perp$ is taking into account the diffraction of the beam.

2.3.4 Gaussian optics

Because of the form of the transverse Laplacian in cylindrical coordinate, the *parabolic* equation (2.41) becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = 2ik\frac{\partial F}{\partial z} \tag{2.43}$$

Actually solving directly this equation is not very easy and therefore we will follow another approach here. Let's assume that the solution has the form of a Gaussian

$$\psi(r) = e^{-i\left[p(z) + \frac{kr^2}{2q(z)}\right]} \tag{2.44}$$

where $p(z)$ is the complex phase with varies with the propagation and $q(z)$ the curvature ($q \in \mathbb{C}$). Note that $\forall r,\ |\psi(r)| = 1$, except if q is a complex number. The main idea here is to find conditions on $p(z)$ and $q(z)$ such that Eq. (2.44) is indeed a solution of the parabolic equation (Eq. 2.43). By introducing the ansazt (eq. 2.44) into (2.43), we obtain the following equation:

$$r^2\frac{k^2}{q^2}\left(\frac{dq}{dz} - 1\right) - 2k\left(\frac{dp}{dz} + \frac{i}{q}\right) = 0 \tag{2.45}$$

[4]

$$\left(\nabla^2 + k^2\right)E = 0 \;\Leftrightarrow\; \left[\partial_{xx} + \partial_{yy} + \partial_{zz}\right]Fe^{-ikz} + k^2Fe^{-ikz} = 0$$
$$\text{and } \partial_{zz}(Fe^{-ikz}) = (\partial_{zz}F)e^{-ikz} - 2ik\partial_z Fe^{-ikz} - k^2Fe^{-ikz}$$

which has a solution for all r if

$$\frac{dq}{dz} = 1 \tag{2.46a}$$

$$\frac{dp}{dz} + \frac{i}{q} = 0 \tag{2.46b}$$

The partial differential eq. (2.45) is now replaced by two independent 1^{st}-order equation.

Complex parameter : *q(z)*

The solution of eq. (2.46a) is straightforward:

$$q(z) = z + \text{const} = z + iZ_R \tag{2.47}$$

Note that the choice of the constant equal to iZ_R is justified by the fact that q cannot be real. If it was, we would have $\forall r$, $|\psi(r)| = 1$, and then the energy is not confined around the optical axis. Then:

$$\psi(r, z = 0) = e^{-ip(0)} \times e^{\frac{-kr^2}{2Z_R}} \tag{2.48}$$

At $z = 0$ the amplitude rapidly decays as we go away from the optical axis. This amplitude is reduced by $1/e$ for $r_0 = \sqrt{2Z_R/k}$. This constitutes an important scaling factor. By naming $r_0 = w_0$ the waist, then

$$Z_R = \frac{\pi w_0^2}{\lambda} \tag{2.49}$$

The length is called the *Rayleigh length*.

q can be expressed as a function of the radius of curvature of the wavefront R and the size of the beam $w(z)$:

$$\frac{1}{q} = \frac{1}{z + iZ_R} = \frac{z - iZ_R}{z^2 + Z_R^2} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w(z)^2} \tag{2.50}$$

and from this, we can express the parameter $w(z)$, which has the same meaning than w_0 at $z = 0$:

$$w^2(z) = w_0^2 \left(1 + \frac{z^2}{Z_R^2}\right) = w_0^2 \left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right] \tag{2.51}$$

We can now see that in eq. (2.44), the term

$$\exp\left(-i\frac{kr^2}{2q(z)}\right) = \exp\left(-ik\frac{r^2}{2R}\right) \times \exp\left(-\frac{r^2}{w^2}\right)$$

is the product of one phase term and one attenuation. The quantity $kr^2/2R$ represents the dephasing of the Gaussian beam. From eq. (2.50), we have the expression of the radius of curvature of the wavefront:

$$R(z) = z\left(1 + \frac{Z_R^2}{z^2}\right) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] \tag{2.52}$$

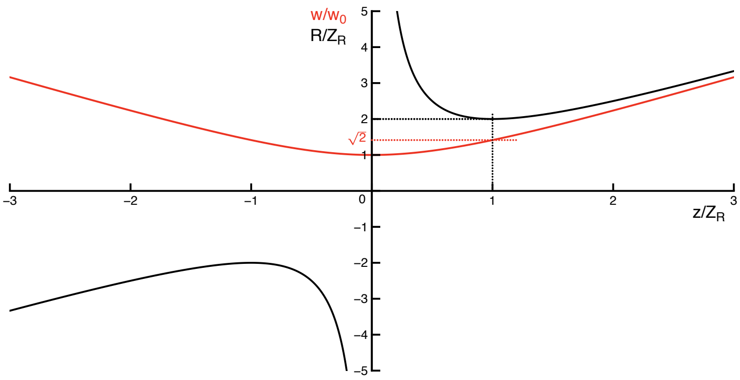

Figure 2.9: Evolution of the radius of the beam normalised with respect to the beam waist w_0 (red) and evolution of the radius of curvature of the wavefront normalised with respect to the Rayleigh length Z_R (black). At the waist ($z = 0$) the wavefront has an infinite radius of curvature $R \to \pm\infty$.

The function $w(z)$ is a hyperboloid, for which the asymptotics lines are tilted by θ with the optical axis. θ measures the divergence of the beam:

$$\tan\theta = \lim_{z\to\infty} \frac{w(z)}{z} = \frac{\lambda}{\pi w_0} \tag{2.53}$$

Complex phase: *p(z)*

Let now integrate the eq. (2.46b). Knowing q, we can write:

$$\frac{dP}{dz} = -\frac{i}{q(z)} = \frac{-i}{z + iZ_R} \tag{2.54}$$

This is an equation with the typical form u'/u, which integrates as a ln function:

$$-ip(z) = -\ln\left(1 - \frac{iz}{Z_R}\right) \tag{2.55}$$

Then

$$e^{-ip(z)} = \frac{1}{1 - i\dfrac{z}{Z_R}} = \frac{1}{\sqrt{1 + \dfrac{z^2}{Z_R^2}}}\, e^{i\varphi} \tag{2.56}$$

with $\varphi = \operatorname{atan}\dfrac{z}{Z_R} = \operatorname{atan}\dfrac{\lambda z}{\pi w_0^2}$. All together the phase shift for the Gaussian beam implies three different terms:

$$\phi(r, z) = kz + \frac{r^2}{2R} - \operatorname{atan}\left[\frac{\lambda z}{\pi w_0^2}\right] \tag{2.57}$$

In comparison with plane wave where there is only the kz phase, we have here two more contributions: a radial and a longitudinal [atan (z/Z_R)] ones. This longitudinal contribution is called the **Gouy phase shift**. Finally the field is described by:

$$E(x, y, z) = \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2}\right) \exp\left(-i\phi\right) \tag{2.58a}$$

$$\text{with } \phi = kz - \operatorname{atan}\left(\frac{z}{z_R}\right) + \frac{kr^2}{2R(z)} \tag{2.58b}$$

2.3.5 Summary

$$\frac{1}{q} = \frac{1}{R(z)} - \frac{i\lambda}{\pi\ w^2(z)}$$	q-parameter	(2.59a)
$$w(z) = w_0\sqrt{\left(1+\frac{z^2}{Z_R^2}\right)}$$	waist	(2.59b)
$$R(z) = z\left[1+\left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$$	radius of curvature	(2.59c)
$$Z_R = \frac{\pi w_0^2}{\lambda}$$	Rayleigh length	(2.59d)
$$\theta = \frac{\lambda}{\pi w_0}$$	divergence	(2.59e)

And λ is the wavelength in the material where the beam is considered!

2.4 How to handle the Gaussian beam?

At the beginning of this chapter we saw that in the *small angles* approximation, we can use ABCD matrix in order to describe the propagation of ray optics through optical elements. Since the Gaussian beam rely on the paraxial approximation, which is very similar it would be good to be able to use the matrices derived for the formalism of ray optics, but in the case of Gaussian optics. As we already mentioned in the case of laser cavity we will only require a very few numbers of transfer matrix.

For free-space propagation the evolution of the complex parameter is absolutely trivial. Looking at the eq. (2.47), it is clear that we have

$$q_2 = q_1 + z \tag{2.60}$$

2.4.1 Transformation by a thin lens

In the situation of the propagation through a thin lens the situation is less obvious and we should look at it a little more carefully. In the present case we use a bi-convex lens (refractive index n), thickness d_0 and both faces have the same radius of curvature $\mathcal{R}$. What is clear however is that the waist of the beam $w(z)$ is not modified by the thin lens. The lens only modifies the curvature of the wavefront $R(z)$, which means that the lens only modifies the phase of the beam!

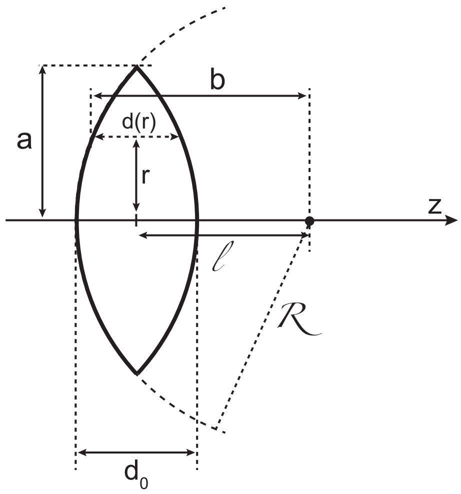

Figure 2.10: bi-convex lens

For a thin lens we have $d_0 \ll a$ and $d_0 \ll \mathcal{R}$. With the present variables we have $\ell^2 = \mathcal{R}^2 - a^2 = (\mathcal{R} - d_0/2)^2$. Then we can write:

$$a^2 = \mathcal{R} d_0 - \frac{d_0^2}{4} \approx \mathcal{R} d_0 \tag{2.61}$$

Let $d(r)$ be the thickness at a distance r from the optical axis ($r \ll \mathcal{R}$). Then, since $b^2 = \mathcal{R}^2 - r^2$, we have:

$$\begin{aligned} d(r) &= d_0 - 2(\mathcal{R} - b) = d_0 - 2\left(\mathcal{R} - \sqrt{\mathcal{R}^2 - r^2}\right) \\ \Leftrightarrow\ d(r) &= d_0 - 2\mathcal{R}\left[1 - \sqrt{1 - \frac{r^2}{\mathcal{R}^2}}\,\right] \\ \Leftrightarrow\ d(r) &\sim d_0 - \frac{r^2}{\mathcal{R}} = \frac{a^2 - r^2}{\mathcal{R}} \end{aligned} \tag{2.62a}$$

The field after the lens has experienced a phase shift so that

$$E_2 = E_1 \exp\left[-ik\delta(r)\right] \tag{2.63}$$

where the effective length between two planes z_1 and z_2 at a distance r from the optical axis is $\delta(r)$. This optical effective length corresponds to a free path over $(d_0 - \delta(r)$ and the path through the lens –$d(r)$– which has a refractive index n. We can then write:

$$\delta(r) = (n-1)d(r) + d_0 \tag{2.64}$$

and so the field after the lens is:

$$E_2 = E_1 \exp\left[-ik(n-1)\frac{a^2 - r^2}{\mathcal{R}} - ikd_0\right] \tag{2.65}$$

Since the focal of a lens is

$$\frac{1}{f} = \frac{2(n-1)}{\mathcal{R}} \tag{2.66}$$

we have:

$$E_2 = E_1 \exp\left[-ik\left(\frac{a^2 - r^2}{2f} + d_0\right)\right] = E_1 \exp\left(\frac{ikr^2}{2f}\right) \exp\left[-ik\left(d_0 + \frac{a^2}{2f}\right)\right] \tag{2.67}$$

We remind that in the paraxial approximation, the field is given by:

$$E(x,y,z) = \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2}\right) \exp\left(-i\phi\right) \tag{2.68a}$$

$$\text{with } \phi = kz - \text{atan}\frac{z}{z_R} + \frac{kr^2}{2R(z)} \tag{2.68b}$$

We can then deduce from eq. 2.68 and eq. 2.67 that:

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{2.69}$$

With the definition of the q-parameter

$$\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2} \tag{2.70}$$

we deduce that:

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \tag{2.71}$$

2.4.2 The ABCD law for Gaussian optics

It is now time to merge the results that we just obtained for the evolution of a Gaussian beam propagating either in free-space or through a thin lens with the transfer matrices that we used in the formalism of ray optics.

	Ray-optics	Gaussian optics
free-path	$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$	$q_2 = q_1 + d$
thin lens	$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$	$q_2 = \dfrac{q_1}{-\frac{1}{f}q_1 + 1}$

Comparing the result of the evolution of the complex parameter q and the transfer matrices of the simple element, we can immediately write the ***ABCD law***:

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}} \tag{2.72}$$

2.5 Laser modes

The fundamental solution of the Helmholtz equation has the form:

$$E_{00}(x, y, z) = \frac{w_0}{w(z)}\, e^{-\frac{r^2}{w^2}}\, e^{-i\left(kz - \varphi + \frac{kr^2}{2R}\right)} \tag{2.73}$$

where the Gouy phase is given by $\varphi = \arctan \frac{z}{Z_R}$. This would correspond to the fundamental mode TEM_{00}. In fact there exist also higher-order modes. For these modes the symmetry of the system must be taken into account. We distinguish the Hermite-Gauss modes (for rectangular symmetry) from the Laguerre-Gauss modes which appears in system with cylindrical symmetry.

2.5.1 Hermite-Gauss modes

In this case the electric field has the form:

$$E_{n,m}(x, y, z) = \frac{w_0}{w(z)}\, e^{-\frac{r^2}{w^2}}\, H_n\left(x\frac{\sqrt{2}}{w}\right)\, H_m\left(y\frac{\sqrt{2}}{w}\right)\, e^{-i\left(kz - \Phi + \frac{kr^2}{2R}\right)} \tag{2.74}$$

with the phase-shift $\Phi = (m + n + 1)\varphi$. Note that each modes has its own phase-shift! (n, m) are two integers representing the variations of the transverse field in x (resp. y) direction. H_n are the Hermite's polynomials which are solution of the differential equation:

$$\ddot{y} - 2x\dot{y} + 2ny = 0$$

A useful way to evaluate the Hermite polynomials is to follow the recursive definition:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

The first few polynomials are[5]

$$H_0(x) = 1 \qquad H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x$$

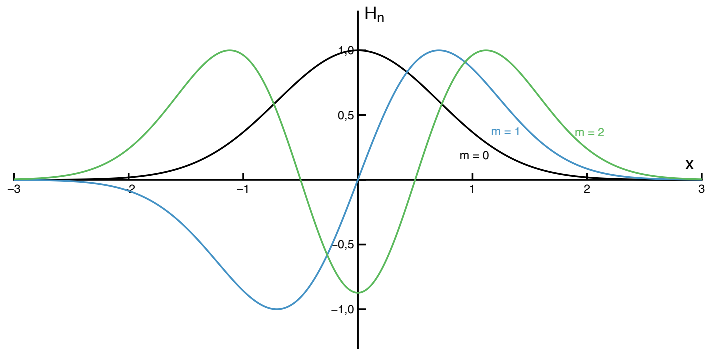

Figure 2.11: $H_n(x)e^{-x^2/2}$ for $n = 0,1,2$

[5]Note that depending of the normalisation that is used, we can also find in the literature the following polynomials:

$$H_0(x) = 1$$
$$H_1(x) = x$$
$$H_2(x) = -x^2 + 1$$
$$H_3(x) = -\frac{2}{3}x^3 + x$$

but then the recursive relation is given by $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$

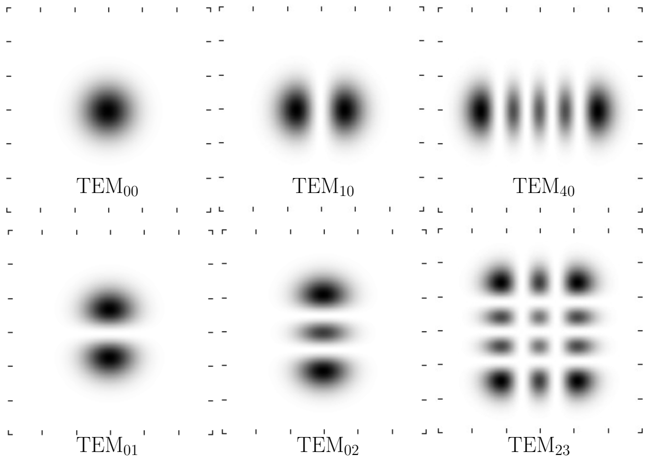

Figure 2.12: Different Hermite-Gauss profile (rectangular symmetry)

2.5.2 Laguerre-Gauss modes

Such modes appear when the system has a cylindrical symmetry. In this case the field can be given by:

$$E_{\ell,p}(r,\theta,z) = \frac{w_0}{w(z)}\, e^{-\frac{r^2}{w^2}} \left(\sqrt{2}\frac{r}{w}\right)^{\ell} L_p^{\ell}\left(\frac{2r^2}{w^2}\right)\, e^{-i\left(kz-\Phi+\frac{kr^2}{2R}\right)} \times \begin{cases} \sin \ell\theta \text{ if } i=1 \\ \cos \ell\theta \text{ if } i=2 \end{cases}$$

In this equation ℓ is the azimuthal index (or angular index) and p is the radial index. They correspond to the number of zeros appearing azimuthally for $\theta \in [0,\pi[$ and radially. For $\ell = 0$, we can write $E_{p0i} = E_{p0}$. Moreover we have $\Phi = (2p+\ell+1)\varphi$. From the general form the Laguerre polynomials can put express as:

$$L_p^{\ell}(x) = \frac{x^{-\ell}}{p!} e^x \frac{\partial^p}{\partial x^p}\left(x^{p+\ell}e^{-x}\right) \tag{2.76}$$

as the first polynomials are

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x+1 \\ L_2(x) &= \frac{x^2-4x+2}{2} \\ L_3(x) &= \frac{-x^3+9x^2-18x+6}{6} \end{aligned}$$

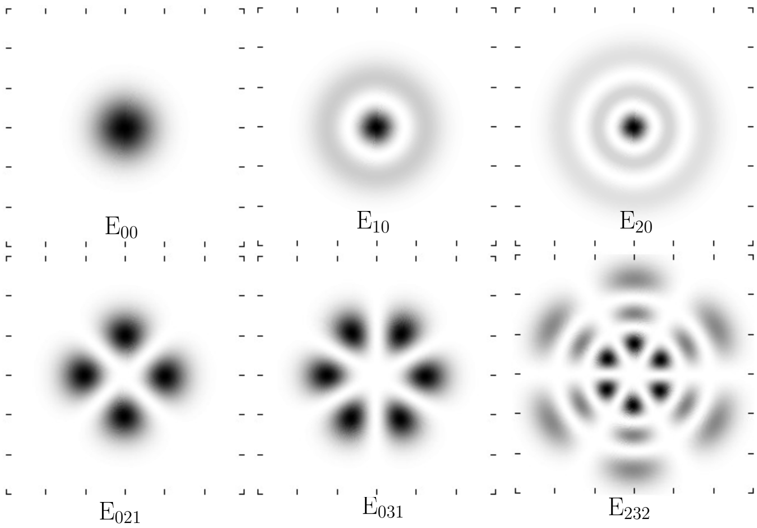

Figure 2.13: Different Laguerre-Gauss profile (circular symmetry)