Advanced laser
INTRODUCTION

0.1 Introduction

Although the first laser was demonstrated fifty years ago, there exist a lot of applications such as:

- Spectroscopy
- Surgery
- Velocimetry
- Communication
- etc.

The goal of this lecture is not to present all these applications, but to try to answer both following questions:

- How to generate beam of light which has wavelength, power, and spatio-temporal characteristics that can be perfectly controlled?
- Why do we manufacture so many types of lasers since we already have very intense light source, which are cheaper?

Actually, the properties of a laser are unique, and make this source much better than any others. Among others, we can cite:

- Spectral range
- Coherence
- Directivity

0.1.1 Laser - the Acronym

The acronym Laser was first introduced by Gordon Gould in 1959. It stands for:

Light **A**mplification by **S**timulated **E**mission of **R**adiation

0.1.2 History of Laser and main applications

1917:	Einstein – stimulated emission (prediction)
1950:	A. Kastler – proposes the inversion of population by optical pumping
1954:	C.Townes – first MASER (NH3 ; 23GHz)
1958:	Townes and Shallow – theory for the MASER in the optical range
1960:	Maiman – first LASER (Ruby)
1961:	Javan – 1^{st} gas laser
	Hellwarth– proposal of Q-switch
	Fraunken– Second Harmonic Generation SHG
1962:	Semiconductor laser
	Stimulated Raman
1964:	Active mode-locking
	Stimulated Brillouin
1966:	sub PS
1969:	SC generation
1982:	Pulse compression
1986:	Ti:Sa
2001:	5fs at 800nm
	250as by HHG
2008:	80as generated by HHG

0.2 Basics ingredients

Any laser requires 3 basics elements:

0.2.1 Optical amplification

The active medium (crystal, gas, liquid) will amplify the incoming electromagnetic field (Fig. 1).

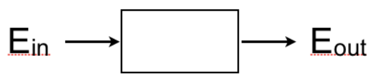

Figure 1: Amplification of the incoming field E_{in}.

During the amplification, the incoming field $E_{in} = E_0 \cos(\omega t)$ is amplified such that[1]

$$E_{out} = \sqrt{G} E_0 \cos(\omega t + \Phi)$$

The is a well defined phase relation between E_{in} and E_{out}. This is equivalent to electronic amplification. G is the gain in intensity.

[1] in complex notation, this equation is written as

$$E_{out} = \sqrt{G} E_{in} e^{i\phi}$$

0.2.2 Feedback loop

Most of the time, the gain is not sufficient to achieve amplification. Intrinsic losses must then be compensated by a feedback loop. This optical system made of mirrors re-injects part of the output radiation at the input of the amplification. This set of mirror forms an optical cavity (Fig. 2).

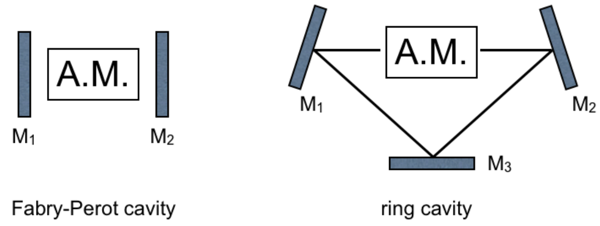

Figure 2: Different kinds of laser cavities. A.M. is the active medium.

Note that the optical cavity not only allows a feedback, useful to balance the intrinsic loss of the system, but also provides resonant frequencies. As we will see the final spectrum of the laser not only depends on the spectral range of the amplification given by the active material itself, but also on the resonant frequencies of the cavity itself.

0.2.3 Output coupling

If we want to have access to the generated radiation, we need an output coupler which has a transmission $T \neq 0$. A laser is then an open system with constant losses: this is a *dissipative* system. The stationary regime is obtained when loss and gain are balanced. This requires to bring energy from the outside, which can be performed electrically (diode), or optically. Finally, it is worth mentioning that the output coupler is not unidirectional, and the intra-cavity field is usually very sensitive to any influence from outside the cavity (*e.g.* back reflection in the laser cavity).

0.3 Properties of the emitted radiation

Waves generated by a laser are an electromagnetic waves (EM). These are propagating waves mathematically described as

$$E(t,z) = \frac{1}{2} E_0 e^{i(\omega_0 t - kz)} + c.c. \tag{1}$$

where $\omega_0 = 2\pi/T$ and $k = 2\pi/\lambda$. T is the temporal period [s], and λ the wavelength, the spatial period [m]. The wave has a velocity $c = 2.99792458 \times 10^8$ m.s^{-1}, such that after one period T, the wave has travelled a distance $\lambda = c \times T = c/\nu$, where ν is the frequency of the wave [Hz]. For visible light, $\lambda \in [0.4, 0.8]$ μm. We can then calculate the order of magnitude of the frequency:

$$\nu_L = \frac{c}{\lambda} \approx \frac{3 \times 10^8}{0.5 \times 10^{-6}} = 6 \times 10^{14} \text{ Hz} \tag{2}$$

The spectral width ($\delta\nu_L$) of the emitted light can be extremely narrow. Considering only the zero-point fluctuations this can be as low as $\delta\nu_L/\nu_L \sim 10^{-18}$. Of course this is extremely small, and in practise *technical noise* are usually predominantly determining the real linewidth of the laser. Except for semi-conductor laser, which are very small, typical values are within 10-50 kHz. Laser cavities can also be specially designed – stabilized – to limit this bandwidth down to ~ 0.1 Hz.

0.3.1 Properties of the beam

Although the properties of the generated beam will depend on the cavity as well as the active medium, we can give a few common properties and a few order of magnitudes:

1. Divergence : $\theta \approx \lambda/D$ with λ the wavelength and D the diameter of the beam.
2. Density of power : In the case of a single transverse mode, the beam can be focused close to the diffraction limit: $w \approx \lambda$. High density of power can then be achieved.
3. Range of available power:
    - μW to kW in continuous wave (CW) regime
    - a few J in pulsed regime
    - 1-10 TW in a 1-ns pulse (Livermore big laser facility)
4. Ultrashort pulse: 5 fs with Ti:Sa Laser
5. Tunability
6. Conversion efficiency from 10^{-3} to 10%
7. Range of wavelength
    - Laser H_2O: $\lambda = 0.11$ μm
    - Acid Formic Laser: $\lambda = 393$ μm.

0.4 Fundamental mechanisms

Idea: We are going to look at the different mechanisms, which take place in the process of amplification. The simple model (*rate equation*) is based on the exchange of energy between the atoms on the level of energy involved in the process with the radiation inside the cavity. This is not the perfect model, but it introduces the notions of threshold, gain and saturation. Moreover, it can reproduce quite accurately the starting of many lasers (class A and class B). Note that there are many different models.

From quantum mechanics, we know that any material (including the active medium) is made of atoms, molecules, ions, which have discrete levels of energy. Each level is characterized by:

- energy E_i
- degeneracy g_i
- number of atoms in that level (density of population) N_i
- lifetime of the level τ_i, so that the decay rate is $\gamma_i = \dfrac{1}{\tau_i}$

Moreover in an atomic medium at a temperature T, the populations of level 1 and 2 follow the Boltzmann-Gibbs distribution (Fig. 3):

$$\frac{N_1}{N_2} = \frac{g_1 \exp\left[-\frac{E_1}{k_B T}\right]}{g_2 \exp\left[-\frac{E_2}{k_B T}\right]} \tag{3}$$

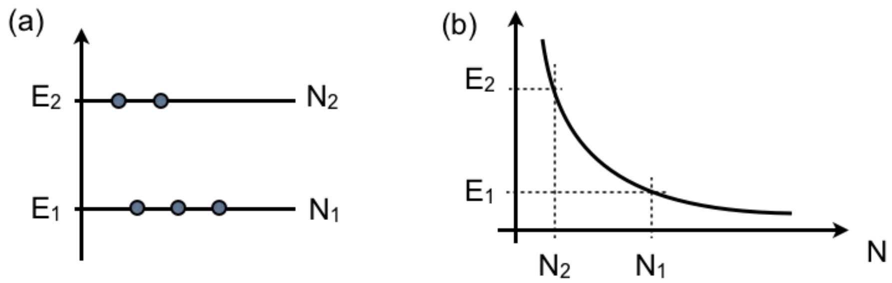

Figure 3: (a) schematic representation of the levels of energy in an atomic medium and (b) the Boltzmann-Gibb distribution of the population for each individual level

When an electromagnetic radiation interacts with the atomic medium, three processes can happen[2]:

1. spontaneous emission
2. absorption
3. stimulated emission

and the photon involved has an energy such that $\Delta E = E_2 - E_1 = h\nu$

0.4.1 Spontaneous emission

In the process of *spontaneous emission*, one atom transits from level 2 to level 1, with the emission of one photon $h\nu$ (Fig 4). During this process the number of atoms on the level 2 decreases (and the population on level 1 increases in the same manner). We can then write[3]:

$$dN_{2,sp} = -\gamma_{||} N_2 dt \tag{4}$$

$\gamma_{||}$ is the decay rate of the level by spontaneous emission[4] and has a dimension of s^{-1}. Of course, since the system is closed, the population on level evolves as

$$dN_{1,sp} = +\gamma_{||} N_2 dt \tag{5}$$

[2] Stimulated emission was first introduced by A. Einstein in 1917.

[3] N_i is the density of atoms of the level i and therefore has the unit of $[m^{-3}]$.

[4] A. Einstein was first to model this phenomenon. He introduces $A_{2\to1}$ or simply A_{21}, the rate of spontaneous emission per atom and per unit of time. The evolution of the number of atoms on level 2 is then

$$dN_{2,sp} = -A_{21} N_2 dt$$

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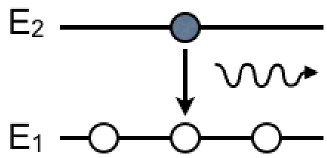

Figure 4: Spontaneous emission

Eq. (4) can easily be integrated: the evolution of the population $N_2(t)$ is simply:

$$N_2(t) = N_2(t=0)\ e^{-\gamma_{\|}t} = N_2(t=0)\ e^{-t/\tau} \tag{6}$$

where τ is the radiative lifetime of the level 2. Table 4 gives the rate of spontaneous emission for few laser media.

Element	wavelength $[\mu m]$	$\gamma_{\|}\ \ [s^{-1}]$
CO_2	10.6	0.3
He-Ne	0.633	800
Nd^{3+}:YAG	1.064	1.4×10^6

Table 1: rate of spontaneous emission

0.4.2 Absorption

During this process, one atom on level 1 gains energy *via* the absorption of one photon $h\nu$, and reached the level 2 (Fig. 5). This phenomenon requires the presence of radiation. The probability to have transition from level 1 to 2 by absorption per atom and per unit of time is proportional to the flux of photons $\mathcal{J}$ and is given by $\sigma_{12}\mathcal{J}$. σ_{12} is the cross section and has the dimension of m^{-2}. It highly depends on the active medium. The

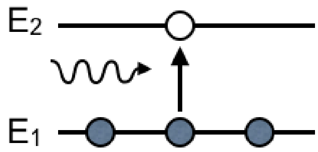

Figure 5: Absorption

population of level 2 increases by[5]

$$dN_{2,abs} = +\sigma_{12}\mathcal{J}N_1dt \tag{7}$$

At the same time, the population on level 1 is reduced by

$$dN_{1,abs} = -\sigma_{12}\mathcal{J}N_1dt \tag{8}$$

[5]Using Einstein coefficients, the probability for an atom to transit from $1 \rightarrow 2$ with absorption of a photon is given by

$$dW_{12} = B_{12}\ u\ dt,$$

where $u(\nu)$ is the spectral power density in power of the electromagnetic radiation. This can be related to the Planck formalism established for black body emission.

0.4.3 Stimulated emission

The concept of *stimulated emission* is not easy to apprehend. Nevertheless this is the key ingredient for any laser. During this process an atom on level 2 transits to the level 1. The desexcitation requires the presence of one photon $h\nu$, which induces the transition. As the atoms transits from $2 \rightarrow 1$, one photon is released (Fig. 6). The generated photon is totally identical to the incident photon (same direction of propagation, same frequency, same state of polarization).

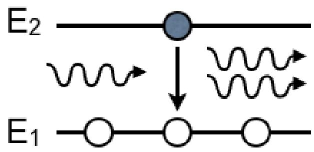

Figure 6: Stimulated emission

During stimulated emission, the flux of photons increases and there is amplification of the incident radiation. The population and the flux of photons evolves as

$$dN_{2,stim} = -\sigma_{12}\mathcal{J}N_2\ dt \tag{9a}$$
$$dN_{1,stim} = +\sigma_{12}\mathcal{J}N_2\ dt \tag{9b}$$

where σ_{12} is the cross section of stimulated emission. From the equation for the absorption and for the stimulated processes, it can be seen that both phenomena are similar. On the other hand spontaneous emission and stimulated emission have fundamentally different properties:

Spontaneous Emission	Stimulated Emission
No photon required to start the process $\rightarrow dN_{sp} \propto N$	Process requires an incident photon $\rightarrow dN_{stim} \propto N \times \mathcal{J}$
The photon created is a random process $\Rightarrow$ Photon generated in any direction	The photon created has the very same characteristics as the inducing photon $\Rightarrow$ **Coherent** process

Table 2: Main characteristics of spontaneous and stimulated emission

0.4.4 Evolution of the flux of photons

To evaluate the variation of the *flux of photons* $\mathcal{J}$ (*i.e.* the number of photons per unit of surface and time $[s^{-1}m^{-2}]$), it is good to note that the number of photons, which go through an area A have all be generated within a volume $Acdt$, where c is the velocity of light (Fig. 7).
For each of the three processes previously described, we can now evaluate their respective effect on the evolution of the flux of photons:

1. Due to the spontaneous emission, there are $\gamma_{\|}N_2$ photons created during dt. This corresponds to a variation of the flux of photons:

$$d\mathcal{J}_{sp} = \gamma_{\|}N_2 c dt \tag{10}$$

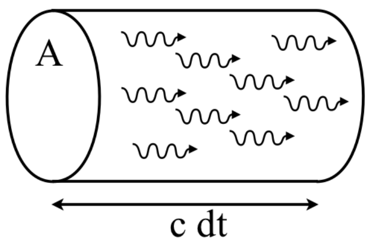

Figure 7: Flux of photons. Every photon generated during dt will cross the area A during dt. They are all in a volume $Acdt$.

2. Due to absorption $\sigma_{12}\mathcal{J}N_1$ atoms on level one have absorbed photon to reach the level 2. Therefore:
$$d\mathcal{J}_{abs} = -\sigma_{12}\mathcal{J}N_1 cdt \tag{11}$$

3. Finally in a duration dt, $\sigma_{12}\mathcal{J}N_2$ atoms from the level 2 decay to the level one with emission of a twin photon due to stimulated emission. The contribution to the flux of photons is then:
$$d\mathcal{J}_{stim} = +\sigma_{12}\mathcal{J}N_2\ cdt \tag{12}$$

0.5 Rate equations

If both levels have the same degeneracy, then cross section of absorption and of emission are equal:
$$\sigma_{12} = \sigma_{21} = \sigma \tag{13}$$

0.5.1 Two-level system

Let us consider a closed system with only two levels. Populations on each level are N_1 and N_2 (Fig. 8) and $N_1 + N_2$ is constant. (Eq. (7), (8), (9a)) and (9b), Without taking

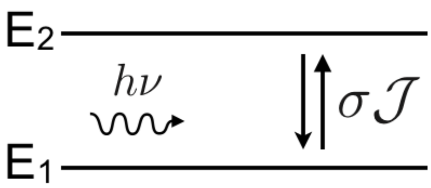

Figure 8: two-level system. Spontaneous emission is not taken into account.

into account any spontaneous decay from level 2 to 1, according to the evolution of the populations and the flux of photons, we can write

$$\dot{N}_2 = -\sigma\mathcal{J}N_2 + \sigma\mathcal{J}N_1 \tag{14a}$$
$$\dot{N}_1 = \sigma\mathcal{J}\left(N_2 - N_1\right) \tag{14b}$$
$$\dot{\mathcal{J}} = c\sigma\mathcal{J}\left(N_2 - N_1\right) \tag{14c}$$

From Eq. (14c), we see that we will have an increase of the flux of photons, if and only if $N_2 > N_1$. However, because of the Boltzmann-Gibbs distribution (Eq. (3)), the population on level 2 cannot exceed the level 1's.

Note that with the spontaneous emission of level 2, we would write

$$\dot{N}_2 = -\gamma_{||} N_2 + \sigma \mathcal{J} (N_1 - N_2) \quad (15a)$$

$$\dot{N}_1 = +\gamma_{||} N_2 - \sigma \mathcal{J} (N_1 - N_2) \quad (15b)$$

And since the system is closed, then $N_1 + N_2$ is constant. At the steady state, we will have

$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0$$

This leads to

$$\begin{cases} N_2(t \to \infty) & \sim \quad \dfrac{\sigma \mathcal{J}}{\gamma_{||} + 2\sigma \mathcal{J}} N_{tot} \\ N_1 & \to \quad N_{tot} - N_2(t \to \infty) \end{cases} \quad (16)$$

If $\mathcal{J} \gg \gamma_{||}/\sigma$ then $N_2 \to 1/2$ and $N_1 \to 1/2$, and if $\mathcal{J} \to 0$ then $N_2 \to 0$. In any case, we will always have $N_2 < N_1$. At best they may equalize: we cannot have any gain in a two level system!

0.5.2 Optical pumping

The optical pumping was first introduced by Alfred Kastler. This is an essential process for the laser as it brings the atom on an upper level, and allows the inversion of population. Let us consider the following system, where we include the pumping λ_i. Note that usually there is only one pumping toward level 2, but it may not be a selective process. If we

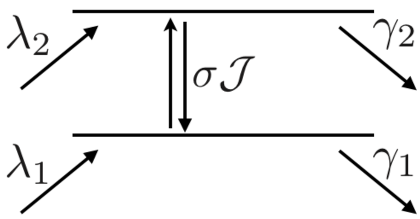

Figure 9: Introduction of the pumping process λ_i.

suppose that there is neither absorption nor stimulated emission, then the evolution of the population on the level i is

$$\dot{N}_i = \lambda_i - \gamma_i N_i \quad (17)$$

This is readily solved as

$$N_i(t) = \frac{\lambda_i}{\gamma_i} \left(1 - e^{-\gamma_i t}\right) \quad (18)$$

$\widehat{N_i}$ is the maximum population than can be reached in the presence of pumping.

0.5.3 Modeling

Taken into account the spontaneous and the stimulated processes we can now write the complete system described by Fig. 9. The evolution of the population and of the flux of

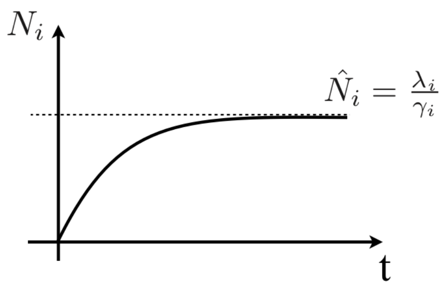

Figure 10: Evolution of the population when pumping and natural decay are the only processes taken into account.

photons are:

$$\dot{N}_2 = -\gamma_{\|} N_2 + \lambda_2 - \sigma \mathcal{J} (N_2 - N_1) \tag{19a}$$

$$\dot{N}_1 = -\gamma_{\|} N_1 + \lambda_1 + \sigma \mathcal{J} (N_2 - N_1) \tag{19b}$$

$$\dot{\mathcal{J}} = \sigma c \mathcal{J} (N_2 - N_1) - k \mathcal{J} \tag{19c}$$

Note that we introduced here the loss of the photons with the term $-k\mathcal{J}$ in eq. (19c). We can introduce the new variables $\Delta N = N_2 - N_1$ is the inversion of population, and $\widehat{\Delta N} = \widehat{N_2} - \widehat{N_1} = \lambda_2/\gamma_2 - \lambda_1/\gamma_1$. The system (19) can then be written as

$$\dot{\Delta N} = -\gamma_{\|} \left(\Delta N - \widehat{\Delta N} \right) - 2\sigma \mathcal{J} \Delta N \tag{20a}$$

$$\dot{\mathcal{J}} = \sigma c \mathcal{J} \Delta N - k \mathcal{J} \tag{20b}$$

Let us consider the stationary solution of this system. This is defined by $\dot{\mathcal{J}} = 0$ and $\dot{\Delta N} = 0$. In the case of no emission of any radiation ($\mathcal{J} = 0$), we find from Eq. (20a) that $\Delta N = \widehat{\Delta N}$. In the case of emission, Eq. (20b) gives the inversion of population at threshold:

$$\Delta N_{th} = \frac{k}{\sigma c} \tag{21}$$

We can rewrite the system (19), introducing the dimensionless inversion of population $D = \Delta N / \Delta N_{th}$, and the pumping rate $A = \widehat{\Delta N} / \Delta N_{th}$:

$$\dot{D} = -\gamma_{\|}(D - A) - 2\sigma D \mathcal{J} = \gamma_{\|} \left[A - D \left(1 + \frac{2\sigma \mathcal{J}}{\gamma_{\|}} \right) \right] \tag{22a}$$

$$\dot{\mathcal{J}} = k \mathcal{J} (D - 1) \tag{22b}$$

Finally we introduce the intensity of saturation so that

$$I = \frac{\mathcal{J}}{\mathcal{J}_{sat}} = \frac{2\sigma \mathcal{J}}{\gamma_{\|}}$$

The model is then expressed by

$$\dot{D} = \gamma_{\|} \left[A - D(1 + I) \right] \tag{23a}$$

$$\dot{I} = kI(D - 1) \tag{23b}$$

The only remaining dimension in these equations is the time. We can easily re-scale this either by respect with k[6] or with the radiative lifetime of the level $\gamma_{\parallel}$:

$$\left.\begin{array}{rcl}\tau & = & \gamma_{\parallel} t \\ \kappa & = & \dfrac{k}{\gamma_{\parallel}}\end{array}\right\} \Longrightarrow \boxed{\begin{array}{rcl} I' & = & \kappa\, I(D-1) \\ D' & = & A - D(1+I)\end{array}} \tag{24a}$$

$$\left.\begin{array}{rcl}\tau & = & kt \\ \gamma & = & \dfrac{\gamma_{\parallel}}{k}\end{array}\right\} \Longrightarrow \boxed{\begin{array}{rcl} I' & = & I(D-1) \\ D' & = & \gamma\,[A - D(1+I)]\end{array}} \tag{24b}$$

In these equations, the *prime* denotes the derivation with respect to τ. These sets of equations are known as the *rate-equations*, and were first introduced by Tang, Statz and deMars[7]. As we can see from the table, although the range of the characteristic times (relaxation and lifetime of the photons in the cavity) are rather different, the ratio $\gamma = \gamma_{\parallel}/k$ remains a small quantity, typically of the order of 10^{-3}. In the case of microchip laser, this could even go down to $\approx 10^{-6}$. Such properties, shared by many laser, is actually responsible for a rather weak stability, which can then be advantageously exploited as we will see later in this lecture. Concerning the pumping rate, it is typically in the range $1-10$.

Note that are other ways to non-dimensionalize the rate equation. One advantage to use the eq. (24b) comes for the small value of γ, which is in factor of the whole equation for the inversion of population, and therefore mathematically convenient.

Laser	k $[s^{-1}]$	$\gamma_{\parallel}$ $[s^{-1}]$	$\gamma = \gamma_{\parallel}/k$
CO_2	9.6×10^6	3×10^4	3.125×10^{-3}
solid-state (Nd^{3+}:YAG)	6.6×10^7	4.1×10^3	6.3×10^{-4}
Semiconductor (AsGa)	10^{12}	10^9	10^{-3}

Table 3: Characteristic times for common lasers

0.6 Steady states

Steady states of the system of dimensionless rate equations

$$I' = I(D-1) \tag{25a}$$
$$D' = \gamma\,[A - D(1+I)] \tag{25b}$$

must satisfy the conditions $\left({}^{dI}/_{d\tau}\right) = \left({}^{dD}/_{d\tau}\right) = 0$, which correspond to two possible solutions:

$$\text{(a) Laser off} \left\{\begin{array}{rcl} I & = & 0 \\ D & = & A\end{array}\right. \qquad \text{(b) Laser on} \left\{\begin{array}{rcl} I & = & A-1 \\ D & = & 1\end{array}\right. \tag{26}$$

Since the intensity cannot be a negative value, we can add that $I = A - 1 \geq 0$, which means that the only possibility to have lasing operation requires $A > 1$. Fig. 11 shows

[6] we remind that $1/k$ is the lifetime of the photon in the cavity.

[7] C.L.Tang, H. Stats and G. DeMars, "*Spectral output and spiking behavior of solid-state lasers*" J. Appl. Phys. **34**, 2289 (1963)

the solutions as a function of the pumping rate A, which is in fact the only accessible parameter remaining in the set of eq. (25). Such diagram is called a *bifurcation diagram* because it represent the possible amplitude for the solutions (here I and D) as a function of a control parameter. For a particular value of this control parameter, the system changes from one state to the other. This happens at the *bifurcation point*[8].

For $A < 1$ (below threshold), the only possible steady state corresponds to the non-lasing operation ($I = 0$). Above the threshold ($A > 1$), there exists two possible solutions. The stability of these solution will actually determine which one will be observed experimentally.

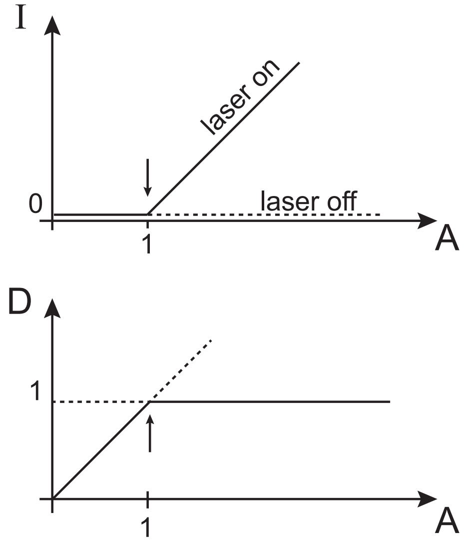

Figure 11: Steady state solutions. Solid (dashed) line indicates the stable (unstable) solution. The arrow indicates the bifurcation point at the lasing threshold $A = 1$.

[8]Study of bifurcations would require a full lecture by itself. We will only see such diagrams occasionally. However a very didactic approach to nonlinear dynamics and bifurcation diagram can be read in the book "Nonlinear dynamics and chaos" from S.H. Strogatz.

0.7 Three level system

In the following system, we will consider that we have a closed system, then the sum of the population is constant. The simplest model that can be imagined is a three level system. Two models are then possible. They are actually a good description of a laser such as ruby laser working at 690 nm (Fig. 12a) or a CO_2 laser (Fig. 12b). For these systems the pumping rate is indicated by W_p, and the relaxation of the levels by γ_{ij}. They are measured in $[s^{-1}]$.

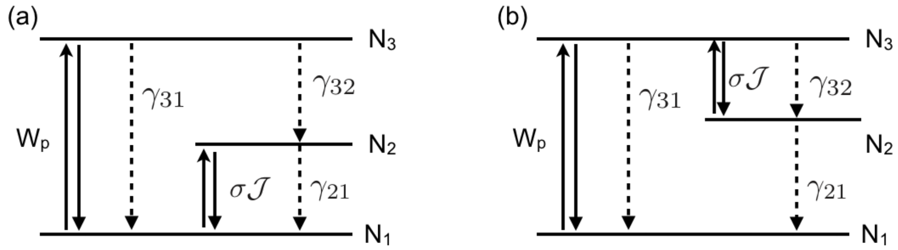

Figure 12: Schematic representation of a three level system. W_p is the pumping rate and γ_{ij} the relaxation between level i and j. (a) Ruby laser and (b) CO_2 laser

Here we focus on the case of Ruby laser (Fig. 12a) e can write the evolution of the population on level 3:

$$\dot{N}_3 = W_p N_1 - W_p N_3 - (\gamma_{32} + \gamma_{31}) N_3 \tag{27}$$

For the scheme to work, the population of level 2 needs to grow, but the atoms need to remain on this level long enough to transit to level 1 by stimulated emission. As a consequence, the relaxation from $3 \rightarrow 2$ must be fast, but the relaxation from $2 \rightarrow 1$ slow. We then need $\gamma_{32} \gg W_p, \gamma_{31}$. Moreover it is to be noticed that the optical pumping W_p will excite atoms from level $1 \rightarrow 3$ but can also induce transition from $3 \rightarrow 1$. However if the decay rate from $3 \rightarrow 2$ is sufficiently fast then there will be no atom remaining on level 3. We can then consider that the $W_p N_1$ atoms excited will go on level 2 (level 3 is virtually empty all the time). Finally we have:

$$W_P N_1 = \gamma_{32} N_3 \tag{28a}$$

$$\dot{N}_2 = \gamma_{32} N_3 + \sigma \mathcal{J} \left(N_1 - N_2\right) - \gamma_{21} N_2 \tag{28b}$$

$$\dot{N}_1 = -W_p N_1 - \sigma \mathcal{J} (N_1 - N_2) + \gamma_{21} N_2 \tag{28c}$$

Introducing the inversion of population $\Delta N = N_2 - N_1$, and $N_{tot} = N_1 + N_2$, we can write the complete set of equations:

$$\dot{\Delta N} = -(W_p + \gamma_{21})\Delta N + (W_p - \gamma_{21}) N_{tot} - 2\sigma \mathcal{J} \Delta N \tag{29a}$$

$$\dot{\mathcal{J}} = -k\mathcal{J} + c\sigma \mathcal{J} \Delta N \tag{29b}$$

0.7.1 Steady state

From Eq. (29a), we can extract the maximum inversion of population at the threshold ($\mathcal{J} = 0$) for the steady state ($\dot{D} = 0$ and $\dot{\mathcal{J}} = 0$).

$$\widehat{\Delta N} = N_{tot} \frac{W_p - \gamma_{21}}{W_p + \gamma_{21}} \tag{30}$$

Then at the threshold of the laser, the pumping rate is

$$W_{p,th} = \gamma_{21} \frac{N_{tot} + \widehat{\Delta N_{th}}}{N_{tot} - \widehat{\Delta N_{th}}} = \gamma_{21} \frac{1 + \dfrac{\widehat{\Delta N_{th}}}{N_{tot}}}{1 - \dfrac{\widehat{\Delta N_{th}}}{N_{tot}}} \tag{31}$$

Since $\Delta N_{th} \ll N_{tot}$, we can say that the pumping rate to reach the threshold is $W_{p,th} \sim \gamma_{21}$. From Eq. (29b), we can write the inversion of population at threshold (Eq. (21)) and introduce the inversion of population $D = \Delta N / \Delta N_{th}$. Then Eq. (29a) writes as

$$\left(\frac{1}{W_p + \gamma_{21}}\right) \dot{D} = \frac{N_T}{\Delta N_{\text{th}}} \left(\frac{W_p - \gamma_{21}}{W_p + \gamma_{21}}\right) - D - \frac{2\sigma I D}{W_p + \gamma_{21}} \tag{32}$$

We then have the pumping rate

$$A = \frac{N_T}{\Delta N_{\text{th}}} \left(\frac{W_p - \gamma_{21}}{W_p + \gamma_{21}}\right) \tag{33}$$

and the intensity of saturation

$$\mathcal{J}_{sat} = \frac{W_p + \gamma_{21}}{2\sigma} \tag{34}$$

Finally we can write the model:

$$\boxed{\begin{array}{rcl} \left(\dfrac{1}{W_p + \gamma_{21}}\right) \dot{D} & = & A - D(1 + I) \\ \dot{I} & = & kI(D - 1) \end{array}}$$

0.8 Four level system

Let us consider now a medium for which the atomic levels are distributed as on Fig. 13. This model can describe a Nd:YAG laser working at 1064 nm.

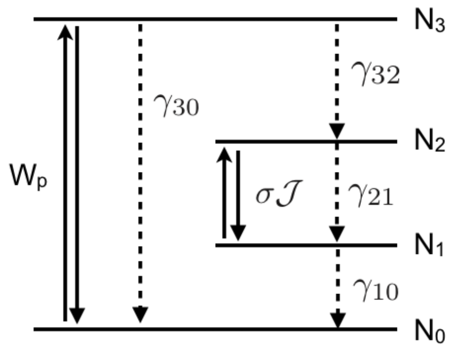

Figure 13: 4-level system. W_p is the pumping rate and γ_{ij} the relaxation decay between the levels i and j.

$$\text{from Fig. 13} \begin{cases} \dot{N}_3 = W_p N_0 - \gamma_{32} N_3 = 0 \Rightarrow \gamma_{32} N_3 = W_p N_0 \\ \dot{N}_2 = N_0 W_p - \gamma_{21} N_2 - \sigma \mathcal{J}(N_2 - N_1) \\ \dot{N}_1 = -\gamma_{10} N_1 + \gamma_{21} N_2 + \sigma \mathcal{J}(N_2 - N_1) \\ N_0 + N_1 + N_2 = N_T \end{cases}$$

with

$$\begin{cases} N_1 + N_2 = N \ \rightarrow N_0 = N_T - N \\ \\ \Delta N = N_2 - N_1 \end{cases}$$

As previously the upper level (here N_3) is empty. We then obtain

$$\begin{cases} N_2 &=& \frac{1}{2}(N + \Delta N) \\ \\ N_1 &=& \frac{1}{2}(N - \Delta N) \end{cases}$$

We can then write:

$$\begin{cases} \Delta \dot{N} = W_p(N_T - N) - (N + \Delta N)\gamma_{21} + \gamma_{10}(\frac{N-\Delta N}{2}) - 2\sigma\mathcal{J}\Delta N \\ \\ \dot{N} = W_p(N_T - N) - \gamma_{10}(\frac{N-\Delta N}{2}) \end{cases}$$

And at steady state $\Delta\dot{N} = 0$, $\dot{N} = 0$, we have

$$\Delta N = \frac{W_p N_T(\gamma_{10} - \gamma_{21})}{\sigma\mathcal{J}(2W_p + \gamma_{10}) + W_p(\gamma_{10} + \gamma_{21}) + \gamma_{10}\gamma_{21}} \tag{35}$$

The system has gain when this quantity is positive, which is automatically satisfied as soon as $\gamma_{10} > \gamma_{21}$. If this is not the case, then the atoms remain excited on level 1, which is bad for the inversion of population $N_2 - N_1$. The maximum inversion of population $\widehat{\Delta N}$ is given by replacing $\mathcal{J}$ by zero in eq. (35). At threshold ($W_P = W_p^{th}$), the Eq. (35) becomes:

$$\widehat{\Delta N}_{th} = \frac{W_p^{th}\ N_T(\gamma_{10} - \gamma_{21})}{W_p^{th}\ (\gamma_{10} + \gamma_{21}) + \gamma_{10}\gamma_{21}}$$

Note that as soon as we have pumping ($W_p > 0$) and $\gamma_{10} > \gamma_{21}$ we have inversion of population. Moreover, in the ideal case where $\gamma_{10} \gg \gamma_{21}$ we can estimate the pumping rate at threshold :

$$W_p^{4\text{ level}} = \frac{\gamma_{21}\widehat{\Delta N}_{th}}{N_T - \widehat{\Delta N}_{th}} \tag{36}$$

We can compare this pumping at threshold with the one for the three level system:

$$\frac{W_P^{4\text{ level}}}{W_P^{3\text{ level}}} = \frac{\widehat{\Delta N}}{\widehat{\Delta N} + N_T} \ll 1 \tag{37}$$

The pumping rate required to reach the inversion of population at threshold is much lower in the case of a 4 level system than for a 3 level system. Note that this model cannot say anything about the maximum power that can be extracted for any of these system.

0.9 Ways to achieve inversion of population

Systems of pumping vary, and to a large extent depend on the type of active medium. Excitation in gas lasers is realized in the simplest manner by means of electric discharge in the active medium (typically glow discharge). In this case the energy of excitation transfers to active centres as a result of collisions with particles in the gas discharge plasma.

As for solid-state lasers, excitation is carried out by irradiating the lasing rod with light from a sufficiently powerful light source: historically were used flash lamp, but nowadays, these are usually pumped by another laser. The active elements are then excited by absorption of one pump photon, and we can see these types of laser as frequency converters, or as a way to convert an incoherent light source (flash-lamp) into a highly coherent laser emission.

0.9.1 solid-states laser

In solid-state laser materials, we need to distinguish the *host matrix* from the active element, which appears actually as an impurity. Both crystal and glass can serve as host matrix. We can cite the sapphire crystal (Al_2O_3) or the yttrium-aluminum garnet ($Y_3Al_5O_12$). The fist glass laser was a fibre laser. Besides Nd^{3+} as an active element, we can find laser based on rare-earth dopant such as Lanthanides (Er^{3+}, Tm^{3+}, Ho^{3+}, and Yb^{3+}). Chrome (Cr^{3+}) and Titanium (Ti^{3+}) are also efficiently used, especially to realize tunable laser or ultrafast source. An *efficient* laser material requires a few properties:

1. The optical pump must be efficiently absorbed by the active element, but not by the host matrix itself.
2. The quantum efficiency must be high: most of the active element getting excited on the upper level of the system must arrive on the upper-level on the lasing transition. It is although important that the loss by other type of transition are limited.

To enhanced these properties, it is common to use rather broad absorption band for the pump. It is also possible to add other elements in order to enhance the absorption of the pump.

Here, we are going to briefly discuss a few solid-state laser, either because of their historical importance, or for the intensive use nowadays. Efficiency up to 20% can be achieved in solid-state lasers.

Ruby laser

This laser was the first to operate. It worked in a pulsed regime. It was demonstrated in 1960 by Theodore Maiman. The ruby is a natural crystal (corundum) doped with chromium ion (Cr^{3+}:Al_2O_3). In general the dopant concentration is around 0.05% (1.6×10^{19} ions/cm^3) up to 0.5%. The fig. 14 shows a simplified energy level for the chromium ion in a ruby laser[9]. In fig. 14b the absorption bands (either to the $^4F_1$ or $^4F_2$ state) are broad and depend on the polarization of the incident beam relative to the optical axis of the crystal. It can be noted that two emission bands of a mercury lamp correspond exactly to these absorption band. This provides a good selectivity of the excitation. After excitation, the active elements fall very rapidly on the metastable states $2A$ and $\bar{E}$. These states are the upper-level of the lasing transition. This scheme can be seen as a 3-level system.

The two bands ($^4F_1$ and $^4F_2$) are connected by a very fast (ps) nonradiative decay to both $2\bar{A}$ and $\bar{E}$ states, which together form the ^2E state. The $2\bar{A}$ and $\bar{E}$ states are themselves connected to each other by a very fast nonradiative decay, which leads to a fast thermalization of their populations, thus resulting in the $\bar{E}$ level being the most heavily populated. Since the total spin of the $2E$ state is $1/2$, the $2E \rightarrow {}^4A_2$ transition is

[9]Notation used to label the levels is derived from group theory.

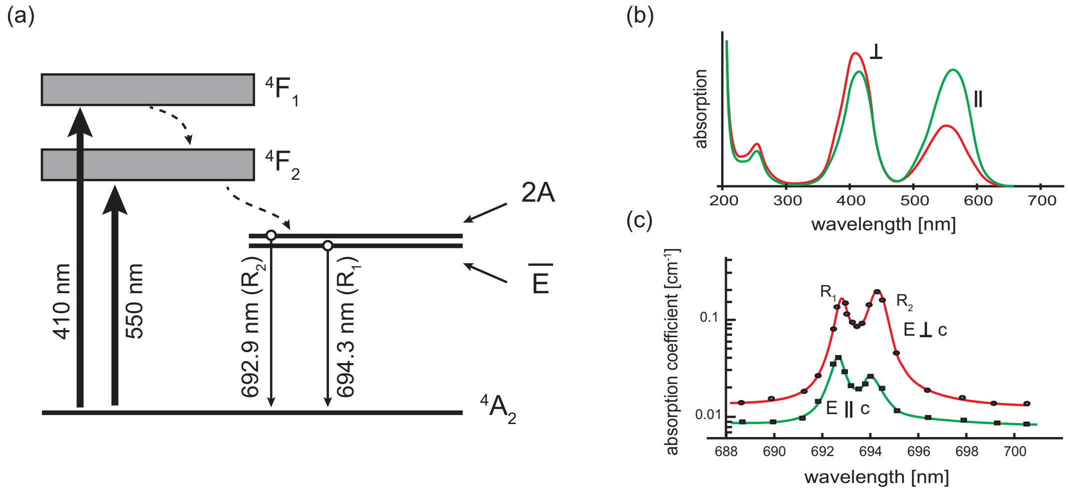

Figure 14: (a) Simplified energy level of the ruby, (b) absorption spectrum depending on the polarization of the incident beam. (c) absorption coefficient of the R_1 and R_2 lines in ruby for an incident beam polarized parallel and orthogonal to the optical axis of the ruby crystal. [from Colin & Webb].

spin-forbidden. The relaxation time of both $2\bar{A}$ and $\bar{E}$ levels to the ground state is thus very long ($\tau \approx 3$ ms), actually one of the longest among all solid-state laser materials. This exceptionally long upper laser level lifetime gives an unusually high-energy storage capability. Since this laser operates on a three-level scheme, the threshold pump energy is typically an order of magnitude higher than that of other solid-state lasers operating with four level schemes (e.g. Neodymium lasers). Moreover due to their long upper state lifetime, ruby lasers lend themselves to Q-switch easily (pulse generation). They can also be used in mode-locked regime.

Ruby crystal are also hard, and have a good thermal conductivity. It can then be used in pulsed regime. Finally it can be grown with excellent optical quality. Ruby lasers, once very popular, are now less widely used since, on account of their higher threshold, they have been superseded by competitors, such as Nd:YAG or Nd:glass lasers. We can however still find them for scientific purpose where their short wavelength compared to Nd:YAG can be an advantage.

Neodymium laser

Neodymium based lasers are certainly among the most popular laser. We can find several matrix host. This is often an yttrium-aluminum garnet crystal ($Y_2Al_5O_{12}$ or YAG), but we can also find fluoride ($YLiF_4$) or vanadate (YVO_4) material and also silicate glass. Note that the neodymium can be replaced by ytterbium ion (Yb^{3+}). Fig. 15 shows a quantum-mechanical energy levels of the Nd^{3+} ion in a Nd:YAG laser crystal (a) as well as a simplified energy level scheme (b)[10].

Nd:YAG lasers can operate either cw or pulsed and can be pumped either by a lamp or by an AlGaAs semiconductor laser. Medium pressure ($500 - 1500$ Torr) Xe lamps and high-pressure ($4 - 6$ atm) Kr lamps are used for the pulsed and cw cases, respectively. If a rod is used as the active medium, the rod diameter ranges typically between 3 and

[10] As for the labelling of the levels, each level is characterized in the form $^{2S+1}L_J$, where S is the total spin quantum number, J is the total angular momentum quantum number, and L is the orbital quantum number. Note that the allowed values of L, namely L = 0, 1, 2, 3, 4, 5, 6, ... are expressed, for historical reasons, by the capital letters S, P, D, F, G, H, I, ... , respectively.

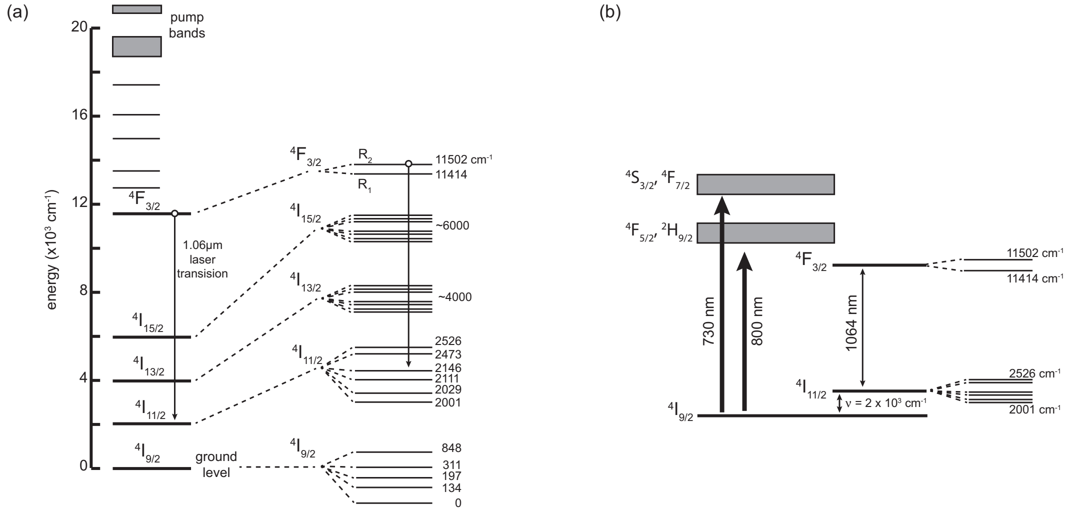

Figure 15: (a) Quantum-mechanical energy levels of the Nd^{3+} ion in a Nd:YAG laser crystal [from Siegman], and (b) simplified energy level [from Svelto]. Upper level splits due to Stark effect (R_1 and R_2).

	Nd:YAG $\lambda = 1064$ nm	$Nd:YVO_4$ $\lambda = 1064$ nm	Nd:YLF $\lambda = 1053$ nm	Nd:glass $\lambda = 1054$ nm (phosphate)
Nd doping [$at.\%$]	1 at. %	1 at. %	1 at. %	3.8 % Nd_2O_3 by weight
$N_t[10^{20}$ ions/cm$^3]$	1.38	1.5	1.3	3.2
τ [μs]	230	98	450	300
$\Delta\nu_0$ [cm^{-1}]	4.5	11.3	13	180
$\sigma_e[10^{-19}$cm$^{-2}]$	2.8	7.6	1.9	0.4
refractive index	$n = 1.82$	$n_o = 1.958$ $n_e = 2.168$	$n_o = 1.4481$ $n_e = 1.4704$	1.54

Table 4: Optical and spectroscopic parameters of Nd:YAG, Nd:YVO4, Nd:YLF, and Nd:glass (phosphate). In the table, N_t is the concentration of the active ions, τ is the fluorescence lifetime, $\Delta\nu_0$ is the transition line-width (FWHM), σ_e is the effective stimulated emission cross section. Data refer to room temperature operation. From [Svelto].

6 mm with a length between 5 and 15 cm. Nd:YAG lasers are widely used in a variety of applications: We can mention:

1. Material processing such as drilling, welding, marking.
2. Medical applications: coagulation and tissue disruption (beam is delivers through optical fiber, inserted into an endoscope.
3. Scientific applications in Q-switch operation. Especially their second ($\lambda = 532$ nm) and third harmonic ($\lambda \approx 355$ nm). Using the second harmonic (intra-cavity), Nd:YAG lasers can nowadays replace Ar-laser, for example for pumping Ti:Sa laser.

Titanium:Sapphire

The most widely used tunable solid-state laser is certainly the titanium sapphire laser. It can operate over nearly $\Delta\lambda \approx 400$ nm ($\Delta\nu \approx 100$ THz). Because of the huge spectral bandwidth, it can also support ultrashort pulses (down to a few fs), and is the workhorse in the race to ultra-short pulse (as). To make Ti:sapphire, Ti_2O_3 is doped into a crystal of Al_2O_3 (typical concentrations range between 0.1 and 0.5% by weight) and Ti^{3+} ions then occupy some of the Al^{3+}-ion sites in the lattice.

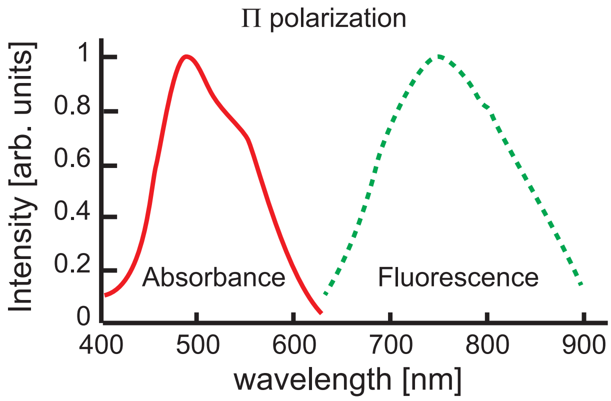

Figure 16: Absorption and fluorescence bands of Ti:sapphire. [from Svelto]

0.9.2 Gas laser

Helium-Neon laser

First gas-laser to be made, the He-Ne laser was demonstrated at Bell Labs by A. Javan and co-workers in late 1960. The first operation was at the transition $2s_2 \rightarrow 2p4$, in the near infrared ($\lambda = 1.15\ \mu$m). Shortly after this demonstration, A.D. White and J.D. Rigden demonstrated the famous red transition at 632.8 nm ($3s_2 \rightarrow 2p4$). As shown on fig. 17, there exists another strong transition at 3.39 μm. There are actually nearly half-dozen possible lasing transition for this system.

The laser tube in a He-Ne laser consists of a few Torr of helium combined with approximately one-tenth that pressure of neon inside a quartz plasma discharge tube sealed with Brewster-angle end windows. To avoid broadening of the laser transition, a mixture of single-isotope He^3 and NE^{20} is usually employed. It is also found that the optimum pressure-diameter product pd in such a laser is a few Torr-mm and that the optimum gain per unit length varies inversely with tube diameter d. The tube is excited with a dc discharge voltage ($\sim 1 \rightarrow 1.5$ kV), producing a dc current typically of order of ~ 10 mA. Free electrons that are accelerated by the voltage collide with helium atoms, which are then excited to the long-lived metastable energy level 2^1S ("2-singlet-S") and 2^3S ("2-triplet-S"). Fortunately, there is a very close energy between each of these helium metastable level and certain sub-levels within the $2s$ and $3s$ groups of excited levels of the neon atoms. At a collision between an excited helium atom with a neon one, the helium can transfer all its energy and get the neon atom in an excited state.

Such laser (especially at the red transition) are very useful as alignment tools, for industrial and scientific purpose, laser scanning microscopy... etc. For the other transitions, laser diode have replaced this laser. At longer wavelength, CO_2 laser is another important laser, whilst in the ultraviolet, we find excimer laser.

CO_2 laser

CO_2 is a molecular laser. This type of laser exploit transitions between the energy levels of a molecule. Depending on the type of transition involved, molecular gas lasers belong

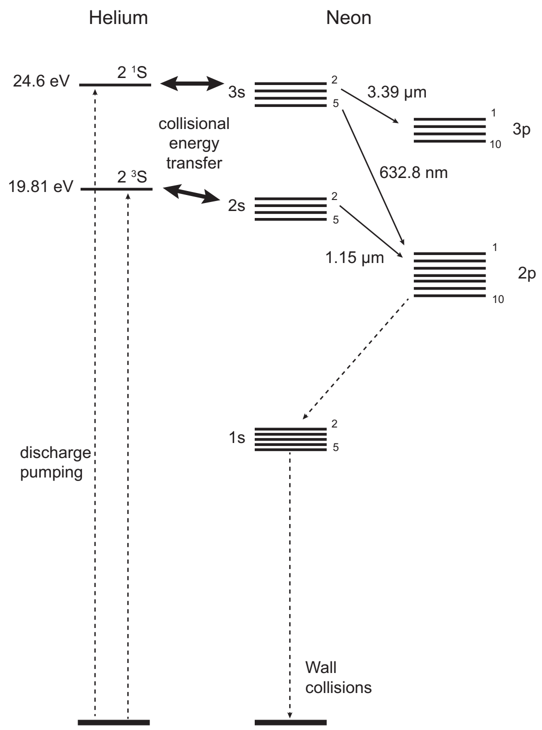

Figure 17: Energy levels in the He-Ne laser

to one of the three following categories:

1. Vibrational-rotational lasers. These lasers use transitions between vibrational levels of the same electronic state (the ground state) and the energy difference between the levels falls in the middle- to the far-infrared ($2.5 \rightarrow 300$ μm). By far the most important example of this category is the CO2 laser oscillating at either 10.6 or 9.6 μm. Other noteworthy examples are the CO laser ($\lambda \approx 5$ μm) and the HF chemical laser ($\lambda \approx 2.7 - 3.3$ μm)

2. Vibronic lasers, which use transitions between vibrational levels of different electronics states: In this case the oscillation wavelength generally falls in the UV region. The most notable example of this category of laser is the nitrogen laser ($\lambda = 337nm$). A special class of lasers, which can perhaps be included in the vibronic lasers, is the excimer laser (*e.g.* ArF or XeCl laser).

3. Pure rotational lasers, which use transitions between different rotational levels of the same vibrational state (usually an excited vibrational level of the ground electronic state). The corresponding wavelength falls in the far infrared (25 μm $\rightarrow$1 mm).

The CO_2 laser is one of the most important lasers for industrial, medical and scientific applications. It is also one of the most powerful laser and output power of 100 kW have been demonstrated for fast gas-flow (CO_2 gas-dynamic laser). Its slope efficiency is also one of the largest ($15 - 20\%$). Applications of CO_2 laser include high-precision material processing, welding and cutting of sheet of metal, surface treatment, tissue coagulation... CO_2 lasers use a gas mixture of helium, nitrogen and carbon dioxide as the active medium, which is usually excited by an electrical gas discharge.

A simplified energy level of the CO_2 laser is shown on fig. 18 (this figure includes the vibrational level of the N_2 molecule, which is used for the pumping. N_2 being a simple diatomic molecule, it only has one vibrational mode, which lowest level are indicated by

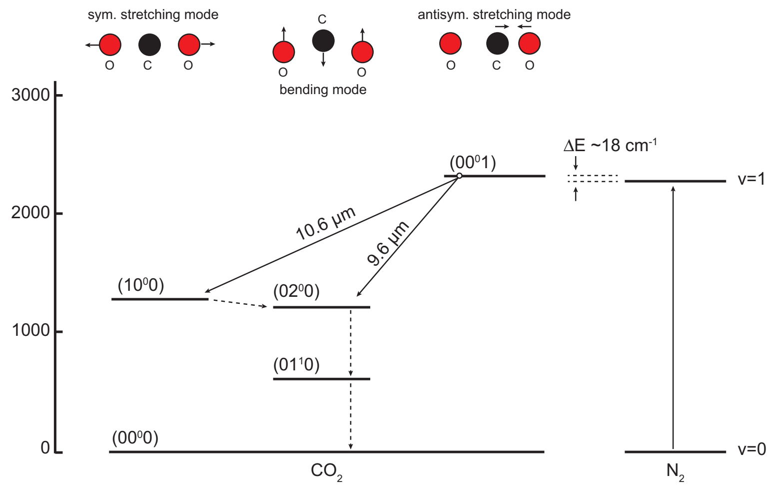

Figure 18: Energy levels in the CO_2 laser

$v = 0$ and $v = 1$. As for the CO_2 molecule, situation is more complex since it is a triatomic molecule, which then exhibits three vibrational mode: (i) a symmetric stretching, (ii) a bending mode and (iii) an anti-symmetric stretching[11].

The pumping of the upper 00^01 laser level is very efficiently achieved by two processes:

1. *Direct Electron Collisions*. The main direct collision to be considered is obviously as follows:
$$e^- + CO_2^{(000)} \rightarrow e^- + CO_2^{(001)}$$

2. *Resonant Energy Transfer from N2 Molecule*. This process is also very efficient due to the small energy difference between the excited levels of the two molecules $\Delta E \approx 18$ cm^{-1}. In addition, the excitation of N_2 from the ground level to the $v = 1$ level is a very efficient process and the $v = 1$ level is metastable.

[11]The oscillation behavior and the corresponding energy levels are therefore described by means of three quantum numbers n_1, n_2 and n_3, which give the number of quanta in each vibrational mode. This means that, apart from zero-point energy, the energy of the level is given by $E = n_1 h\nu_1 + n_2 h\nu_2 + n_3 h\nu_3$, where ν_i is the resonance frequency of the modes i. For example, the 01^10 level corresponds to an oscillation in which there is one vibrational quantum in mode 2. Since mode 2 has the smallest force constant of the three modes (the vibrational motion is transverse), it follows that this level will have the lowest energy. The superscript (which we will denote by ℓ) on the bending quantum number arises from the fact that the bending vibration is, in this case, doubly degenerate.

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