Introduction to entanglement and its application to cavity optomechanics

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The aim of these lectures (and of the present notes) is to provide a short and very practical introduction to entanglement and its evaluation in optomechanical systems. Their first version was prepared for a training workshop held in Erlangen on October 7-9 2013, within the Marie Curie ITN on Cavity Quantum Optomechanics "cQOM". This is an updated version for the first workshop of the new ETN project on Optomechanical Technologies, "OMT".

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I. INTRODUCTION

→ Entanglement describes correlations between two subsystems **stronger then any classical correlation**.

→ Its importance relies on the fact that it allows to perform **tasks which are otherwise impossible in the absence of entanglement**, such as quantum teleportation, quantum dense coding, quantum metrology (that is, measurement of parameters scaling to zero with the number of the systems N, faster then $N^{-1/2}$).

→ There are also tasks which are certainly helped by presence of entanglement but where its is not clear yet: *a*) quantum computation (is the exponential speed-up really due to entanglement ?) *b*) quantum cryptography (quantum key distribution strictly requires only non-orthogonal states)

A. Basic definitions

→ *pure states*: vectors $|\psi\rangle$ of an Hilbert space H.

→ *mixed states*: density matrices ρ: $\rho \geq 0$, $Tr(\rho) = 1$ ⇔all eigenvalues λ_i are such that $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$.

→ *pure state* ⇒ $\rho^2 = \rho$, $\rho =$ orthogonal projector ⇔ only one eigenvalue is $\bar{\lambda} = 1$, all the others are equal to zero. $\rho = |\bar{\psi}\rangle\langle\bar{\psi}|$, $|\bar{\psi}\rangle$= eigenstate with eigenvalue =1.

In these lectures we shall restrict to *bipartite* systems, whose Hilbert space is the tensor product of the two Hilbert spaces of two parties, A and B, $H = H_A \otimes H_B$.

1. Definition for pure states

$|\psi\rangle$ entangled $\Longleftrightarrow$ its Schmidt rank $r > 1$. In fact $|\psi\rangle$ can be always decomposed using the Schmidt decomposition, which is unique (up to some phase)

$$|\psi\rangle = \sum_{j=1}^{r} c_j |\psi_j\rangle_A |\phi_j\rangle_B,\ r = \text{Schmidt rank} = \text{number of terms} \tag{1}$$

$|\psi\rangle$ is not entangled $\Longleftrightarrow$ $|\psi\rangle$ product state = factorized state.
When r=d=dimension of the two subsystems and $|c_j| = 1/\sqrt{d}$, $j = 1, 2...d$, then $|\psi\rangle$ is a *maximally entangled state.*

2. Definition for mixed states

ρ *separable* $\overset{\text{def}}{\Longleftrightarrow}$ $\rho = \sum_{J=1}^{n} P_J \rho_J^A \otimes \rho_J^B$ (=convex sum) $P_J > 0, \sum_{J=1}^{n} P_J = 1$, ρ_J^A and ρ_J^B density matrices of systems A and B. ρ entangled $\Longleftrightarrow$ ρ not separable [1, 2].

In the case of pure states the situation is simple and it is easy to verify if a state is entangled or not. In the case of mixed states instead it is very hard to detect separability or entanglement, and therefore the definition is not operational.

B. Main features of entanglement

Entangled states are not *consistent with local realistic descriptions*. The two parts, A and B, possess properties that cannot be defined locally, i.e., independently from the other party [3]. This is quantified by the violation of a Bell inequality.
It is possible to see that:

- for pure states, violation of a Bell inequality $\Longleftrightarrow$ entangled state.
- for mixed states, violation of a Bell inequality ⇒ entanglement, but the reverse is not true.

The first and well known example is provided by *Werner states* $W(x)$ [1]

$$W(x) = x|\psi_-\rangle\langle\psi_-| + (1-x)\frac{I}{4}$$

where $|\psi_-\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]$ singlet state, and $I = |0\rangle_A\langle 0|\,|0\rangle_B\langle 0| + |0\rangle_A\langle 0|\,|1\rangle_B\langle 1| + |1\rangle_A\langle 1|\,|0\rangle_B\langle 0| + |1\rangle_A\langle 1|\,|1\rangle_B\langle 1| = 4\times 4$ identity matrix ($\rho = I/4$ is the maximally mixed state). Werner proved that

$$x > \frac{1}{3} \Rightarrow W(x)\,\text{entangled},$$
$$x > \frac{1}{\sqrt{2}} \Longleftrightarrow \text{violation of Bell inequalities},$$

and therefore there are mixed bipartite entangled states which does not violate Bell inequalities. It is quite curious

that even though the relevance of entanglement in quantum mechanics with respect to the classical description of phenomena was already pointed out by Schrödinger in 1935, and the work of Bell related to the nonlocal aspects of quantum mechanics, it is only in 1989 that Werner settled the proper definitions and the proper context. In an entangled state, the"local" information is always less then the "nonlocal" information $\Leftrightarrow$ the information of the state is mostly contained in the correlations between the subsystems [4]. Local information is described by the reduced states $\rho_A = Tr_B(\rho)$ and $\rho_B = Tr_A(\rho)$ and it can be quantified by

$$I_A = Tr_A(\rho_A \log \rho_A), \quad I_B = Tr_B(\rho_a \log \rho_B), \quad I_A, I_B \le 0.$$

For example, for a singlet state

$$\rho = |\psi_-\rangle\langle\psi_-|, \qquad I_{AB} = Tr_{AB}(\rho \log \rho) = 0.$$
$$I_A = I_B = -\log 2 < I_{AB} \qquad \rho_A = \rho_B = \frac{I_2}{2} = \text{maximally mixed state}. \tag{2}$$

One defines the *von Neumann entropy* as the opposite of the information

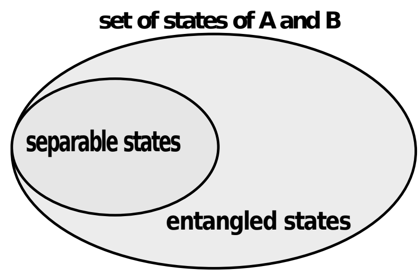
$$S_V(\rho) = -I = -Tr(\rho \log \rho) \tag{3}$$

and it is easy to verify that it is maximum for the maximally mixed states. In fact, in dimension d $\rho_A = \rho_B = \sum_{i=1}^{d} |i\rangle\langle i|/d$ and therefore $S_V(\rho) = \log d$.

II. ENTANGLEMENT (OR SEPARABILITY) CRITERIA

Given ρ they allow to establish if ρ is entangled or not. The set of all bipartite states ρ and the set of separable states *are both convex* (S convex $\Leftrightarrow \rho_1 \in$ S, $\rho_2 \in$ S $\Rightarrow \alpha\rho_1 + (1-\alpha)\rho_2 \in$ S $\forall \;\; 0 \le \alpha \le 1$).)

A. PPT (positive partial transpose) separability criterion



The Horodecki family provided an equivalent definition of separable states [5, 6]:

$$\rho \text{ separable} \Leftrightarrow I_A \otimes \wedge_B(\rho) \ge 0 \;\; \forall \wedge_B,$$

where $\wedge_B$ is a positive map i.e., which preserves the positivity of the density matrix ρ. Also this characterization of separable state is not very much operational because one has to verify it on ALL positive maps, and we do not have a complete decomposition of positive maps.

We note in passing that it is sufficient to verify the above separability conditions only on positive but not completely positive (CP) maps. In fact, by definition, CP map $\Leftrightarrow$ such that any extension in a larger Hilbert space $I_A \otimes I_B$ is positive, therefore for CP maps $\wedge$, the condition of the separability is always satisfied.

However, such a characterization in terms of positive maps provides a necessary condition for separability, and therefore a sufficient condition for entanglement: if $\wedge_B$ positive map exists such that $I_A \otimes I_B(\rho) < 0$, then ρ is entangled.

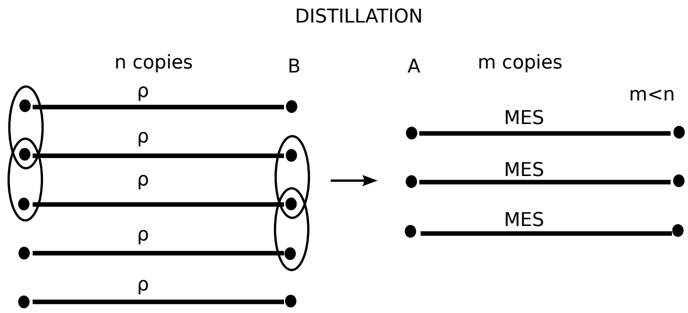
An important positive but not CP map has been found by Peres and Horodecki [5, 7], the **transposition map** $\wedge_b(\rho) = \rho^{T_B}$ (i.e., we evaluate the transposition only on the indices of the subsystem B) and this has led to a well known and very useful **PPT separability criterion**:

$$\rho \text{ separable} \Rightarrow \rho_B^T \geq 0 \tag{4}$$

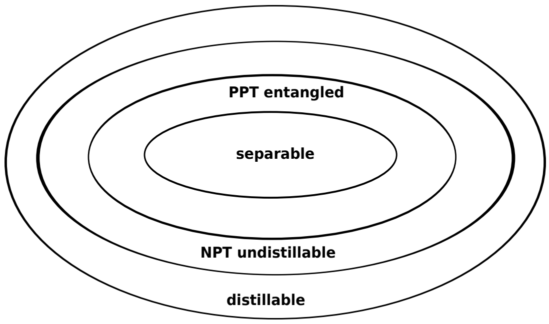
$$\rho^{T_B} < 0 \Rightarrow \rho \text{ entangled (NPT entanglement criterion).} \tag{5}$$

The great success of this criterion is that it allows to detect MANY entangled states, i.e., there are "few" PPT entangled states. In particular: for 2×2 (two qubits) and 2×3 (qubit and qutrit) systems **PPT is necessary and sufficient**, i.e., there is no PPT entangled state. In larger dimensions (2 × 4, 3 × 3 or greater) **PPT entangled states exist**. PPT entangled states are also called ***bound* entangled states**, because in these states entanglement is "locked" and difficult or even impossible to use to perform some tasks.

An example is provided by the **distillation** protocol [8]: starting from n identical copies of a bipartite state ρ, one can "distill" a smaller number $m < n$ of *maximally entangled states* (MES), by using only **local operations** on each part and classical communication of the results of eventual local measurements (LOCC).



It is possible to prove that ρ PPT $\Longrightarrow \rho$ undistillable (i.e., we get zero copies of MES starting from many copies of ρ using only LOCC). The reverse is not yet known in general, that is ρ undistillable $\overset{??}{\Rightarrow}$ ρ PPT state. Therefore in general one cannot exclude the existence of NPT undistillable states, as shown in the figure [2].

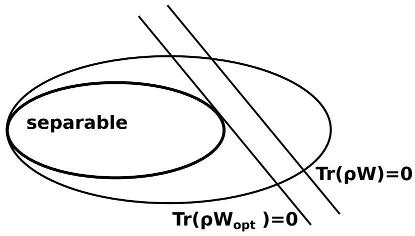


B. Entanglement witnesses

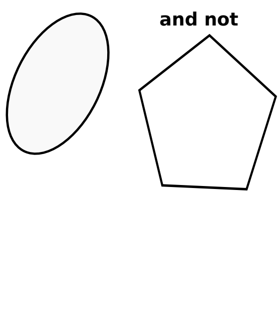
Alternative sufficient conditions to prove that a state is entangled can be obtained using the so called entanglement witnesses:

ρ entangled $\Leftarrow$ $\exists$ hermitian operator W such that

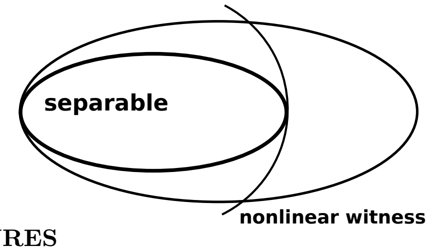
$\langle W \rangle = Tr(\rho W) < 0$ & $\text{Tr}(\rho_{sep} W) > 0 \; \forall \rho_{sep}$ separable state.



Entanglement witness have a simple geometrical interpretation. $Tr(\rho W) = 0$ corresponds to an hyperplane in the state space. Hyperplane tangent to the set of separable state = optimal witness W_{opt}. → The set of separable states is **not** a convex polytope and therefore standard "linear" witnesses W **cannot** provide a complete characterization of the set of separable states [2].



→ Something better can be obtained from "*nonlinear*" witnesses, that is, involving also nonlinear functions of ρ: a simple example is provided by variances, i.e. $\langle \Delta W^2 \rangle = Tr(\rho W^2) - [Tr(\rho W)]^2$, which has a quadratic term in ρ.



III. ENTANGLEMENT MEASURES

They address the questions: (i) how much entangled is ρ ? ii) is ρ_1 more entangled than ρ_2 ?

We normally ask for desirable properties of an entanglement measure; very many measures have been proposed and only few of them possess all the desirable properties; unfortunately these few ones are very difficult to evaluate. Again the more complex and difficult situations occur for mixed states. In fact, if one restrict to pure states only, an acceptable measure of entanglement exists and can be evaluated in terms of the von Neumann entropy of the reduced state. For a pure bipartite state $\rho = \rho^2$

$$E(\rho) = S_v(\rho_A) = S_v(\rho_B), \quad \rho_A = Tr_B(\rho), \ \rho_B = Tr_A(\rho), \tag{6}$$

where $S_v(\rho_J) = -Tr(\rho_J \log \rho_J)$ and $S_v(\rho_A) = S_v(\rho_B)$, because of the Araki-Lieb inequality, $|S_v(\rho_A) - S_v(\rho_B)| \leq S_v(\rho) \leq S_v(\rho_A) + S_v(\rho_B)$. In fact, if ρ is pure then $S(\rho) = 0 \Rightarrow S_v(\rho_A) = S_v(\rho_B)$. If we use the unique Schmidt decomposition of Eq. (1), we get

$$\rho_A = \sum_{j=1}^{r} |c_j|^2 |\psi_j\rangle_A \langle\psi_j|, \ \ \rho_B = \sum_{j=1}^{r} |c_j|^2 |\phi_j\rangle_B \langle\phi_j| \Rightarrow E(\rho) = -\sum_{j=1}^{r} |c_j|^2 \log |c_j|^2. \tag{7}$$

Required properties of an entanglement measure $E(\rho)$:

- ρ separable $\Rightarrow E(\rho) = 0$
- no increase under LOCC $\Leftrightarrow E\left(\wedge_{LOCC}(\rho)\right) \leq E(\rho)$
- local unitary invariance $\Leftrightarrow E(U(\rho)) = E(\rho)$ when $U = U_A \otimes U_B$= local unitary operation
- Continuity: $||\rho_1 - \rho_2|| \to 0 \Rightarrow |E(\rho_1) - E(\rho_2)| \to 0$

Desirable but not strictly required properties:

- Additivity $E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2)$
- Strong super-additivity: given a state ρ acting on the duplicated Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$ with restrictions ρ_i to $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$, if $E(\rho) \geq E(\rho_1) + E(\rho_2)$, with equality if $\rho = \rho_1 \otimes \rho_2$.
- Subadditivity $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$
- Convexity $E(\alpha\rho_1 + (1-\alpha)\rho_2) \leq \alpha E(\rho_1) + (1-\alpha)E(\rho_2)$.

A. Example of entanglement measures

- **Entanglement of formation**

$$E_F(\rho) = \quad \inf \sum_J P_J S_v(|\psi_J\rangle\langle\psi_J|), \tag{8}$$

where the infimum is taken over all possible pure state decomposition of ρ, $\rho = \sum_J P_J |\psi_J\langle\langle\psi_J|$. It is difficult to calculate and it has been shown that it is not additive in general [9].

- **Entanglement cost**

$$E_c(\rho) = \lim_{n\to\infty} \frac{E_F(\rho^{\otimes n})}{n} \quad (\neq E_F(\rho) \text{ in general due to non} - \text{additivity}), \tag{9}$$

$$E_c(\rho) = \min_{\text{LOCC}} (\lim_{N\to\infty} \frac{M_{MES}(N)}{N}), \tag{10}$$

$$\tag{11}$$

where $M_{MES}(N)$= number of maximally entangled states (MES) necessary to create N copies of ρ from them using only LOCC.

- **Entanglement of distillation**

$$E_d(\rho) = \max_{\text{LOCC}} \left(\lim_{N\to\infty} \frac{M^{out}_{MES}(N)}{N} \right), \tag{12}$$

where $M^{out}_{MES}(N)$= number of MES which can be distilled out of N copies of ρ using only LOCC. Also $E_c(\rho)$ and $E_d(\rho)$ are very difficult to calculate.

- **Negativity** Negativity is the quantification of how much the PPT criterion fails

$$N(\rho) = \frac{||\rho^{T_B}|| - 1}{2}$$

$$||\rho^{T_B}|| = Tr\sqrt{\rho^{T_B}\rho^{T_B\dagger}} \quad \Rightarrow N(\rho) = max\left\{0, -\sum_i \lambda_i^-\right\}, \tag{13}$$

where λ_i^- = negative eigenvalue of ρ^{T_B}

- **Logarithmic negativity**

$$E_N(\rho) = \log[1 + 2N(\rho)] = \log||\rho^{T_B}|| \tag{14}$$

$E_N(\rho)$ is the easiest to calculate, it is additive, but it has important drawbacks. In fact, but it is not convex, and it is not continuous [10], so it does not satisfies one of the required conditions. It is in fact considered an entanglement monotone, not a real entanglement measure. An important consequence of the fact that it is not continuous is that it does not always reduce to the unique measure for pure states, entropy of entanglement $S_v(\rho_A)$.

$$E_F(\rho) \geq E_c(\rho) \geq E_d(\rho) \quad \forall \quad \rho \quad \text{and also} \quad E_N(\rho) \geq E_d(\rho). \tag{15}$$

IV. "SYMPLECTIC" DESCRIPTION OF CONTINUOUS VARIABLE SYSTEMS

We now apply these results to the case of continuous variable systems, which naturally applies to optomechanics, which involves optical and mechanical modes. Mechanical modes are continuous variable (CV) systems ("modes") described by

$$\hat{x} = \text{position} = \sqrt{\frac{\hbar}{m\omega_m}} \left(\frac{a + a^\dagger}{\sqrt{2}}\right) = \sqrt{\frac{\hbar}{m\omega_m}} q \tag{16}$$

$$\hat{p} = \text{momentum} = \sqrt{\hbar m\omega_m} \left(\frac{a - a^\dagger}{\sqrt{2}i}\right) = \sqrt{\hbar m\omega_m} p \tag{17}$$

with commutation relation $[\hat{x}, \hat{p}] = i\hbar \Leftrightarrow [q, p] = i$, equivalent to the bosonic commutation relation $[a, a^\dagger] = 1$. Electromagnetic modes are CV systems described by

$$E_\phi = \text{electric field} = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 V}} [ae^{-i\phi} + a^\dagger e^{i\phi}] \tag{18}$$

$$X = \frac{a + a^\dagger}{\sqrt{2}} = \text{amplitude quadrature} \tag{19}$$

$$Y = \frac{a - a^\dagger}{\sqrt{2}i} = \text{phase quadrature,} \tag{20}$$

with commutation relation [X,Y]=i.

In the case of N modes we define the vector of CV operators $u^T = (q_1p_1, q_2p_2, ...q_Np_N)$, with commutation relations $[u_k, u_j] = i\Omega_{kj}$,

$$\Omega = \text{Symplectic matrix} = \bigoplus_l \Omega_l, \quad \Omega_l = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{21}$$

$$\Omega = \begin{pmatrix} \Omega_l & 0 & 0 & 0 & 0 & . & . & . \\ 0 & \Omega_l & 0 & 0 & 0 & . & . & . \\ 0 & 0 & \Omega_l & 0 & 0 & . & . & . \\ 0 & 0 & 0 & \Omega_l & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \Omega_l \end{pmatrix} \tag{22}$$

A quantity of fundamental importance for describing the entanglement of CV systems is the correlation or covariance matrix V

$$V_{kj} = \frac{\langle u_k u_j + u_j u_k \rangle}{2} - \langle u_k \rangle \langle u_k \rangle. \tag{23}$$

The commutation relations put a **strong constraint on V through the Heisenberg relation**

$$V + \frac{i\Omega}{2} \geq 0 \tag{24}$$

generated from the Heisenberg theorem. Intuitively it means that the mode variances “cannot be simultaneously too small”, and therefore V must be ”more than positive”.

Linear canonical transformation are described by $u^{'} = Mu$, with $M = 2N \times 2N$ matrix. In order to be physical, this transformation must preserve commutation rules, which is equivalent to say that the symplectic matrix Ω is preserved, $\Rightarrow M\Omega M^T = \Omega$. Linear canonical transformations are represented by the symplectic group of matrices M preserving the symplectic matrix Ω.

Using symplectic transformations, the Heisenberg condition can be expressed in a different form. In fact, Williamson theorem (the one guaranteeing the existence of usual normal mode decomposition in linear harmonic oscillator systems) proves that V can always be diagonalized by a symplectic M, i.e., $D = MVM^T$.

$$V + i\frac{\Omega}{2} \geq 0 \Rightarrow D + i\frac{\Omega}{2} \geq 0, \tag{25}$$

with $D = \text{diag}(d_1, d_1, d_2, d_2, ...d_N d_N)$, $\quad d_j$ = symplectic eigenvalues.

$$\begin{pmatrix} d_j & \frac{i}{2} \\ -\frac{i}{2} & d_j \end{pmatrix} \geq 0 \Leftrightarrow d_j^2 - \frac{1}{4} \geq 0, \quad d_j \geq \frac{1}{2} \Leftrightarrow V + i\frac{\Omega}{2} \geq 0. \tag{26}$$

The Heisenberg condition is therefore equivalent to the condition that the symplectic eigenvalues of V must be equal or larger than 1/2 (equal to the size of the vacuum fluctuations in the chosen units).

V. ENTANGLEMENT CRITERIA FOR CV SYSTEMS

Let us see first of all how the PPT separability criterion applies to CV systems. Since $\rho = \rho^\dagger$ and $\rho^T = \rho^* \Leftrightarrow$ **transposition** $\Leftrightarrow$ **time inversion**. At the level of CV system therefore $\rho^{T_B} \Leftrightarrow (q_A, p_A, q_B, p_B) \rightarrow (q_A, p_A, q_B, -p_B)$ [11]:

$$u' = \wedge_B u, \quad \wedge_B = \text{diag}(1, 1, 1, -1) \quad \wedge = \wedge^T \tag{27}$$

ρ separable $\Rightarrow \wedge_B V \wedge_B + \frac{i}{2}\Omega \geq 0$. **Therefore, $\wedge_{\mathbf{B}} \mathbf{V} \wedge_{\mathbf{B}} + \frac{\mathbf{i}}{\mathbf{2}}\boldsymbol{\Omega} < \mathbf{0}$ is the NPT sufficient condition for entanglement**. NPT condition is not necessary in general, and therefore we may have CV PPT entangled states. Using again the Williamson theorem for diagonalizing $\tilde{V} = \wedge_B V \wedge_B$ and get the symplectic eigenvalues of $\tilde{V}$ of the partial transposed state, $\tilde{d}_j$

$$\tilde{V} + \frac{i}{2}\Omega < 0 \Leftrightarrow \tilde{d}_j < 1/2. \tag{28}$$

Therefore the NPT sufficient condition for entanglement is easily written in terms of the symplectic eigenvalue $\tilde{d}_j$: $0 < \tilde{d}_j < 1/2$. There exist a simple formula for $\tilde{d}_j$ in terms of V for a bipartite CV system [12]:

$$= \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad A, B, C = 2 \times 2 \quad \text{matrices} \tag{29}$$

$$\tilde{d}_j = \frac{1}{\sqrt{2}}\sqrt{\Sigma(V) - \sqrt{\Sigma(V)^2 - 4\text{Det}V}} \tag{30}$$

$$\Sigma(V) = \text{Det}(A) + \text{Det}(B) - 2\text{Det}(C). \tag{31}$$

The NPT condition $\Leftrightarrow \tilde{d}_j < 1/2$ is equivalent to the inequality $4\text{Det}V < \Sigma(V) - 1/4$.

A. Nonlinear witnesses

There are other sufficient conditions for CV entanglement, given in terms of nonlinear witness,to be more precise in terms of variances of CV quantities.

- Duan-Giedke-Cirac-Zoller sufficient condition ("sum criterion") [13]). For bipartite CV systems

  $$\rho \text{ separable } \Rightarrow \forall\, a \neq 0 \text{ VAR}\left(|a|q_A + \frac{q_B}{a}\right) + \text{VAR}\left(|a|p_A - \frac{p_B}{a}\right) \geq a^2 + \frac{1}{a^2}, \tag{32}$$

  which can be read as: $\exists\, a \neq 0$ such that

  $$\text{VAR}\left(|a|q_A + \frac{q_B}{a}\right) + \text{VAR}\left(|a|p_A - \frac{p_B}{a}\right) < a^2 + \frac{1}{a^2} \Rightarrow \rho \quad \text{entangled}, \tag{33}$$

  where, $\text{VAR}(q) = \langle q^2 \rangle - \langle q \rangle^2 = Tr(\rho q^2) - [Tr(\rho q)]^2$.

- Mancini-Giovannetti-Vitali-Tombesi sufficient condition ("product criterion") [14]

  $$\rho \quad \text{separable} \Rightarrow \forall\, a \neq 0 \text{ VAR}\left(|a|q_A + \frac{q_B}{a}\right) \times \text{VAR}\left(|a|p_A - \frac{p_B}{a}\right) \geq \left(a^2 + \frac{1}{a^2}\right)^2. \tag{34}$$

  It can be read as: $\exists\, a \neq 0$ such that

  $$\text{VAR}\left(|a|q_A + \frac{q_B}{a}\right) \times \text{VAR}\left(|a|p_A - \frac{p_B}{a}\right) < \left(a^2 + \frac{1}{a^2}\right)^2 \Rightarrow \rho \quad \text{entangled}. \tag{35}$$

  This second "product" criterion is easier to satisfy than the sum criterion, i.e., it detects more entangled states than those detected by the "sum" criterion.

VI. THE SPECIAL CASE OF GAUSSIAN CV STATES

Let us now focus on the important special case of Gaussian CV states, which are defined in terms of their phase space description. Such a description consists in associating to a state its Wigner function with phase-space variables, $\rho \leftrightarrow W(u)$. This association is carried out by means of the symmetrically ordered characteristic function:

$$\mathcal{X}_s(\zeta) = Tr[\rho D(\zeta)], \tag{36}$$

where $D(\zeta)=e^{iu^T\Omega\zeta}$ is the displacement operator, $u^T=(q_1,p_1,q_2,p_2,....)$ is the vector of CV phase space operators, $\zeta^T=(\zeta_1,\zeta_2,\zeta_3,....)$ are real variables. The Wigner function is the Fourier transform of this symmetrically ordered characteristic function

$$W(u)=\frac{1}{\pi^{2n}}\int d^{2n}\zeta e^{i\zeta^T\Omega u}\mathcal{X}_s(\zeta), \tag{37}$$

where $u=q_1,p_1,....$ are the real phase-space variables. **A CV state ρ is Gaussian $\Leftrightarrow$ $\mathbf{W(u)}$ is Gaussian $\Leftrightarrow$ $\mathcal{X}_\mathbf{s}(\zeta)$ is Gaussian**, i.e.,

$$\rho \quad \text{Gaussian} \Leftrightarrow \mathcal{X}_s(\zeta)=e^{-\frac{1}{2}\zeta^T V\zeta+id^T.\zeta}, \tag{38}$$

where $V_{ij}=\frac{1}{2}\langle u_iu_j+u_ju_i\rangle-\langle u_i\rangle\langle u_j\rangle$ is the covariance matrix and $d_i=\langle u_i\rangle$ is the mean displacement vector. This latter vector can be varied by applying local displacements, which however do not change the covariance matrix V. Therefore d_i can always be put to zero without affecting the entanglement properties of the state, which are therefore fully determined by the covariance matrix V only. This fact is valid for Gaussian states only, where all the higher order moments can be expressed in terms of the first order moments d_j and second order moments V_{ij}. Non-Gaussian CV states instead depends also upon on higher-order moments.

When $d_j=0$, the Wigner function of a Gaussian state of an N-mode CV system is given by

$$W(u)=\frac{1}{(2\pi)^N\sqrt{\text{Det}V}}e^{-\frac{1}{2}u^TV^{-1}u}. \tag{39}$$

Let us count the number of independent parameters of the CM V of an N-modes system.

V is a $2N\times 2N$ symmetric matrix $\Rightarrow 2N(2N+1)/2$ independent real elements. $3N$ parameters are *local* because each mode is characterized by two phases and a local squeezing parameter r, $q\to rq$ and $p\to p/r$. Therefore an N-modes Gaussian state is characterized by $N(2N+1)-3N=2N(N-1)$ nonlocal parameters. In the case of $N=2$ modes, four nonlocal parameters fully characterizes a Gaussian state. If one restrict to pure states, there are additional constraints and it turns out that pure states depend only upon 1 nonlocal parameter. In fact, in the case of pure Gaussian state the following theorem holds [15]:

Any pure bipartite Gaussian state can always be transformed by local Gaussian transformations (i.e. rotations, phase shifts and single mode squeezing) into a TWO-MODE SQUEEZED STATE (TMS) ρ_{TMS}

$$\rho_{TMS}=|\psi\rangle_{TMS}\langle\psi| \Leftrightarrow V_{TMS}=\frac{1}{2}\begin{pmatrix}\cosh 2r & 0 & \sinh 2r & 0\\ 0 & \cosh 2r & 0 & -\sinh 2r\\ \sinh 2r & 0 & \cosh 2r & 0\\ 0 & -\sinh 2r & 0 & \cosh 2r\end{pmatrix}, \tag{40}$$

where $|\psi\rangle_{TMS}=e^{r(a_1^\dagger a_2^\dagger-a_1a_2)}|0\rangle$ in terms of the annihilation operators a_1 and a_2 of the two modes. Therefore the unique nonlocal parameter of a pure Gaussian states is the squeezing parameter r. $\rho_{\mathbf{TMS}}$ **is the paradigmatic example of a CV entangled state, allowing to show the physical meaning of CV entanglement**. In fact, in such a state,

$$VAR(q_1-q_2)=V_{11}+V_{33}-2V_{13}=e^{-2r} \tag{41}$$

$$VAR(p_1+p_2)=V_{22}+V_{44}+2V_{24}=e^{-2r} \tag{42}$$

$$VAR(q_1+q_2)=VAR(p_1-p_2)=e^{2r}. \tag{43}$$

When $r\to\infty$, ρ_{TMS} tends to the simultaneous eigenstate of the commuting variables. q_1-q_2 and p_1+p_2 which is the famous Einstein-Podolsky-Rosen (EPR) state with its EPR correlations. This means that CV entanglement is always characterized by EPR correlations between linear combinations of position, momentum and field quadratures. These EPR correlations allow to infer the value of a quadrature at one party from the knowledge of a quadrature of the other party.

A. Important theorem for Gaussian bipartite CV states

The PPT criterion is necessary and sufficient for $\mathbf{1\times N}$ CV Gaussian states, i.e., one party has N modes and the other party has 1 mode only, [16]

$$\rho \quad \text{Gaussian} \quad 1\times N \quad \text{bipartite separable} \quad \Leftrightarrow \wedge_B V\wedge_B+i\frac{\Omega}{2}\geq 0 \tag{44}$$

$$\rho \quad \text{Gaussian} \quad 1\times N \quad \text{bipartite entangled} \quad \Leftrightarrow \wedge_B V\wedge_B+i\frac{\Omega}{2}< 0. \tag{45}$$

This means that Gaussian PPT entangled (bound entangled) states exists. 2×2 Gaussian bound entangled states have been found theoretically and also demonstrated experimentally [17]. Therefore for $1 \times N$ Gaussian bipartite CV states, the condition

$$d_j < 1/2 \Leftrightarrow 4\text{Det}V < \Sigma(V) - 1/4 \tag{46}$$

is a necessary and sufficient condition for entanglement. That is, they detect all Gaussian entangled states.

Connection with Duan-Giedke-Cirac-Zoller sum criterion In the case of 1×1 Gaussian bipartite CV states, the necessary and sufficient entanglement condition $\tilde{d}_- < 1/2$ can be written using only appropriate variances, i.e., can be written in the form of a sum criterion. In fact,

$$\tilde{d}_- < 1/2 \Leftrightarrow \text{VAR}\left(aq_A - \frac{c_1}{|c_1|}\frac{q_B}{a}\right) + \text{VAR}\left(ap_A - \frac{c_2}{|c_2|}\frac{p_B}{a}\right) < a^2 + \frac{1}{a^2}, \tag{47}$$

with, however, **specific choices of** $\mathbf{a, c_1, c_2}$ in terms of the matrix elements of the CM V, when it is written in a specific *standard* form

$$V = \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{pmatrix}, \quad \text{with } a^2 = \sqrt{\frac{m_1 - 1}{n_1 - 1}} = \sqrt{\frac{m_2 - 1}{n_2 - 1}},$$
$$|c_1| - |c_2| = \sqrt{(n_1 - 1)(m_1 - 1)} - \sqrt{(n_2 - 1)(m_2 - 1)}. \tag{48}$$

This standard form can always be obtained with appropriate local Gaussian operations (rotations, phase shifts and local squeezing) on the two modes.

This Duan et al. form of the necessary and sufficient condition for entanglement $\tilde{d}_j < 1/2$ is instructive because it points out a general aspect of Gaussian CV entanglement: **in a CV entangled state one can always find a pair of linear combinations of "quadratures",** $\mathbf{v_1 = (aq_A - \frac{c_1}{|c_1|}\frac{q_B}{a})}$ **and** $\mathbf{v_2 = (ap_A - \frac{c_2}{|c_2|}\frac{p_B}{a})}$ **with sufficiently small variance, which therefore possess EPR-like correlations**. In the case of pure Gaussian states we have seen this fact with the theorem showing the equivalence with the two-mode-squeezed state ρ_{TMS}. In the more general case of mixed states, this equivalence with ρ_{TMS} is not perfect, because the two linear combinations in general do not commute, being $[v_1, v_2] = (a^2 + \frac{c_1 c_2}{|c_1 c_2| a^2})i$. Since in an entangled state, $c_1 c_2$ it can be seen that it is always negative, this case occurs only if $a = \pm 1$, which is equivalent to have a **symmetric** Gaussian bipartite state, because $a^2 = 1 \Leftrightarrow m_1 = n_1$ and $m_2 = n_2$.

VII. ENTANGLEMENT MEASURES IN BYPARTITE GAUSSIAN CV SYSTEMS

One can apply the generic proposed entanglement measure (E_F, E_c...) to the case of CV systems but, due to the infinite dimension, their calculation is even more difficult. Again, the case of Gaussian CV system is easier and as, we will see below, it is also relevant for optomechanics where often one operates in the so-called linearized regime. In practice two entanglement measures can explicitly calculated and therefore are mostly used in the Gaussian bipartite case, (E_N, E_F .

A. Logarithmic Negativity

:

$$E_N(\rho) = \max\left\{0, -\log\left(\sum_i 2\tilde{d}_i^-\right)\right\} \tag{49}$$

$\tilde{d}_i^-$ are the symplectic eigenvalues which violate the Heisenberg condition, that is all those such that $\tilde{d}_i^- < 1/2$. If $\tilde{d}_i^- \to 0$ the entanglement increases. A useful formula, in the simple case of 1×1 bipartite CV states, provides the physical meaning of this symplectic eigenvalue and of the log negativity,

$$2\tilde{d}^- = \min_{LOCC} \frac{\text{VAR}(q_A - q_B) + \text{VAR}(p_A + p_B)}{2}, \tag{50}$$

where $\min_{LOCC}$ means that the minimum is taken over all possible local operations on A and B (that is, rotations, phase shifts, squeezing). This means that the PT symplectic eigenvalue $\tilde{d}^{-}$ gives the minimum sum of EPR variances, i.e., $2\tilde{d}_{-} \sim e^{-E_N}$, **so that we can see the logarithmic negativity E_N as a sort of generalization of the squeezing parameter $2r$ of the two-mode squeezed state**. However we will see that this parallel between logarithmic negativity and two-mode squeezing parameter cannot be pushed too far. Therefore, even though most papers use E_N, one has reasons to use another entanglement measure in the Gaussian bipartite case.

This is due to an important result **the extremality of Gaussian quantum states** found by Wolf-Giedke-Cirac [19]: **Given a state ρ (generally non-Gaussian) with CM V, and the Gaussian state ρ_G with the same CM V, one has**

$$\mathbf{E}(\rho) > \mathbf{E}(\rho_{\mathbf{G}}), \tag{51}$$

that is, the Gaussian state provides a lower bound for the entanglement of the state, provided that we use an appropriate entanglement measure $E(\rho)$, which satisfies two conditions: i) it is continuous; ii) it is strongly superadditive. The logarithmic negativity E_N is neither continuous nor strongly superadditive and cannot be used to provide such a lower bound for the entanglement. It means that its value can be either greater ar larger than the entanglement of the state ρ whose entanglement we want to estimate and therefore its value does not provide useful information in such a case.

B. Entanglement of Formation

Very recently it has been found that there is a computable alternative which does not suffer this problem in the Gaussian bipartite case: **Entanglement of formation E_F**. It is in fact continuous; moreover even though it is general not additive, and therefore not strongly superadditive [9], **in the restricted class of two-mode Gaussian states it is both additive [20] and also strongly-superadditive [21]**. More explicitly, Ref. [20] (see also Simon [18]) found that the minimization over all possible convex decompositions in the definition of E_F can be restricted only over Gaussian states, i.e. E_F coincides with the *Gaussian* E_F, allowing to derive an explicit formula for $E_F(\rho)$ for bipartite Gaussian states,

$$E_F(\rho) = \cosh^2 r_o \log(\cosh^2 r_o) - \sinh^2 r_o \log(\sinh^2 r_o). \tag{52}$$

There is still an analytical difficulty because r_o is an effective squeezing parameter $r_o > 0$ that must be determined as a solution of an involved nonlinear system of equations [18, 20], which depend upon the matrix elements of V. However, once that one has found such optimal r_o, one has a proper entanglement measure, which reproduces the pure state entanglement measure, provides a lower bound for every non-Gaussian state with the same V, and also coincides with the entanglement cost E_c because of additivity which holds in this case.

This latter expression also provides also a physical meaning for E_F: In fact, any covariance matrix V of a two-mode Gaussian state can be decomposed as $V = V_{\text{pure}}(r) + V_{\text{noise}}$, where $V_{\text{pure}}(r)$ refers to a pure Gaussian state (with two-mode squeezing r therefore) and V_{noise} is a positive matrix. Eq. 52 is just the von Neumann entropy of the minimally entangled pure state of the above decomposition, that is, the optimal r_o corresponds to the optimal pure TMS state which provides the decomposition $V = V_{\text{pure}}(r) + V_{\text{noise}}$ with the smallest possible squeezing. Therefore **E_F quantifies the mimimum amount of squeezing needed to prepare an entangled state**.

More recently Tserkis and Ralph [22] have provided further progress in this direction. In fact, they simply notice that, given the optimal decomposition $V = V_{\text{pure}}(r_o)+V_{\text{noise}}$, if one applies an anti-squeezing operation with parameter $-r_o$ one gets a factorized state. This provides two interesting results: i) first of all a computable formula for r_o, because it must be that the resulting mimimum symplectic eigenvalue must be 1/2, that is, $2\tilde{d}^{-}[S(-r_o)V(r_o)S(r_o)] = 1$; ii) we have a further physical interpretation of E_F: **it quantifies the minimum amount of anti-squeezing needed to disentangle an initially entangled state**. The explicit computable solution for r_o is

$$r_o = \frac{1}{2}\ln\sqrt{\frac{\kappa - \sqrt{\kappa^2 - \lambda_+\lambda_-}}{\lambda_-}} \tag{53}$$

where $\kappa = 4\text{Det}V + 1/2 - (a-b)^2$, $\lambda_\pm = \Sigma(V) + 2[(ab - c_1c_2) \pm (c_1 - c_2)(a+b)]$ [22, 23], where $\Sigma(V)$ is given by Eq. (31) and a, b, c_1, and c_2 refer to the canonical *normal form* of the covariance matrix V,

$$V = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}. \tag{54}$$

This formula is a significant advance for the calculation of r_o and therefore of E_F. It is convenient to re-express it in terms of local symplectic invariants only, so that one does not have to put the V in hands in the above normal form. The quantities Det$V$ and $\Sigma(V)$ are symplectic invariants; also $a = \sqrt{\mathrm{Det}A}$ and $b = \sqrt{\mathrm{Det}B}$, and $c_1 c_2 = \mathrm{Det}C$ are symplectic invariants. It is also possible to see that also the quantity $c_1 - c_2$ is a symplectic invariant and can be expressed as $c_1 - c_2 = \sqrt{[\mathrm{Det}C - \sqrt{\mathrm{Det}A\mathrm{Det}B}]^2 - \mathrm{Det}V/2(\mathrm{Det}A\mathrm{Det}B)^{1/4}}$ [24]. For what concerns the physical meaning, only E_F can be associated explicitly with squeezing and the above statements about the minimum amount of squeezing anti-squeezing can be applied only to E_F and not to E_N.

VIII. CALCULATION OF ENTANGLEMENT IN OPTOMECHANICAL SYSTEMS

It is quite easy to evaluate entanglement whenever the optomechanical problem can be mapped into a Gaussian problem. This can be done whenever the linearized treatment of the optomechanical problem can be safely assumed. In fact, the noises can almost always assumed to be Gaussian, and therefore linear dynamics subject to Gaussian noise ⇒ Gaussian states of the optomechanical system. Therefore in these cases, in order to calculate entanglement, one has simply to calculate the covariance matrix V [25]. Linearized dynamics ⇒

$$\dot{u}(t) = Au(t) + n(t) \tag{55}$$

where $A =$ drift matrix, $u = (q, p, X, Y)^T$, and $n(t) =$ noise vector, yielding the formal solution

$$u(t) = M(t)u(0) + \int_0^t dsM(s)n(t-s), \quad M(t) = e^{At}, \tag{56}$$

$$V(t) = M(t)V(0)M(t)^T + \int_0^t dsM(s)DM(s)^T \tag{57}$$

where V(t) is the time-dependent covariance matrix, and D is the diffusion matrix giving the correlation between the noise terms,

$$\frac{\langle n_i(t)n_j(s) + n_j(s)n_i(t)\rangle}{2} = D_{ij}\delta(t-s). \tag{58}$$

In particular one can consider the steady state, i.e. the state achieved asymptotically at long times when the system is stable, that is when $M(t) \to 0$ for $t \to \infty$:

$$V_\infty = \int_0^{+\infty} dsM(s)DM(s)^T. \tag{59}$$

V_∞ can be evaluated also in other two ways. For example transforming the time integral into a frequency integral using the Fourier transform. In fact,

$$M(\omega) = \int dtM(t)e^{i\omega t} = \int dte^{At}e^{i\omega t} = -(A + i\omega)^{-1}, \tag{60}$$

which, inserted into the time integral, finally yields [25]

$$V_\infty = \int \frac{d\omega}{2\pi}(A + i\omega I)^{-1}D(A^T - i\omega I)^{-1} \tag{61}$$

The solution V_∞ in terms of the time integral, when the system is stable is also equivalent to the solution of the so called Lyapunov equation

$$AV + VA^T = -D. \tag{62}$$

[1] R. Werner, Phys. Rev. A **40**, 4277 (1989).
[2] D. Bruss, J. Math. Phys. **43**, 4237 (2002).

[3] J. S. Bell, Physics **1**,195200 (1964).
[4] E. Schrödinger, Naturwissenschaften **23**, 807 (1935).
[5] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[6] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81** 865942 (2009).
[7] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[8] C. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[9] M. B. Hastings, Nature Phys. **5**, 255 (2009).
[10] M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).
[11] R. Simon, Phys. Rev. Lett.**84**, 2726 (2000).
[12] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004); G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[13] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[14] S. Mancini, V. Giovannetti, D.Vitali, P. Tombesi, Phys. Rev. Lett. **88**, 120401 (2002); V. Giovannetti, S. Mancini, D.Vitali, P. Tombesi, Phys. Rev. A **67**, 022320 (2003).
[15] G. Giedke, J. Eisert, J. I. Cirac, and M. B. Plenio, Quant. Inf. Comp. **3**, 211 (2003).
[16] R. F. Werner, M. M Wolf, Phys. Rev. Lett. **86**, 3658 (2001).
[17] J. DiGuglielmo, A. Samblowski, B. Hage, C. Pineda, J. Eisert, and R. Schnabel, Phys. Rev. Lett. **107**, 240503 (2011).
[18] J. Solomon Ivan and R. Simon, arXiv:0808.1658v1.
[19] M. M. Wolf, G. Giedke, and J. I. Cirac, Phys. Rev. Lett. **96**, 080502 (2006).
[20] P. Marian and T.A. Marian, Phys. Rev. Lett. **101**, 220403 (2008).
[21] A. A. Pomeransky, Phys. Rev. A **68**, 032317 (2003).
[22] S. Tserkis and T. C. Ralph, arXiv:1705.03612.
[23] There is some difference between the explicit expressions used here and that of Ref. [22], because in the latter they use differently scaling variables, $[q_k, p_j] = 2\mathrm{i}\delta_{kj}$.
[24] G. Adesso, PhD Thesis, University of Salerno, (2006).
[25] C. Genes, A. Mari, P. Tombesi, and D. Vitali, Phys. Rev. A, **78** 032316 (2008).