Reminder wave dynamics:

Bloch waves are extended

$$\psi_{n,\vec{k}}(\vec{x}+\vec{a}) = e^{i\vec{k}\cdot\vec{a}}\,\psi_{n,\vec{k}}(\vec{x})$$

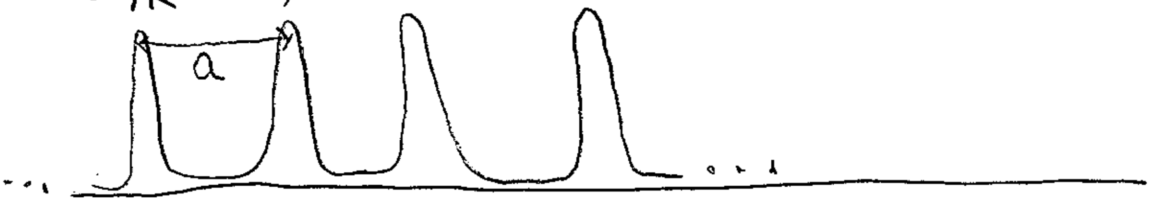
Bloch solutions can also be written in the form:

$$\psi_{n,\vec{k}}(\vec{x}+\vec{a}) = e^{i\vec{k}\cdot\vec{x}}\,\phi_{n,\vec{k}}(x)$$

where $\phi_{n,k}(\vec{x}+\vec{a}) = \phi_{n,\vec{k}}(\vec{x})$

Example (in 1D)

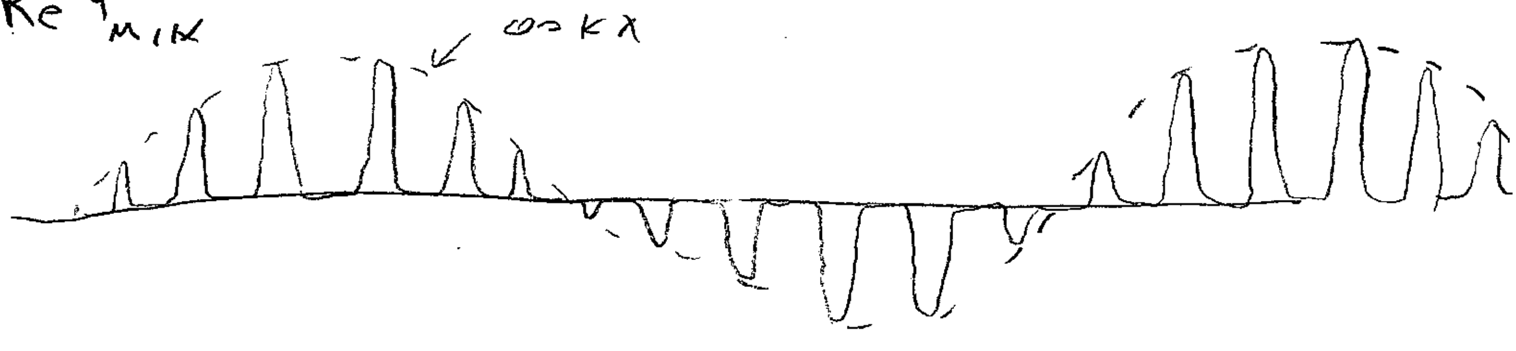
$\phi_{n,k}(\vec{x})$



- could describe Bloch-wave for a chain of atoms where the electrons are strongly localized about nuclei
- In optical settings, Bloch wave for a chain of cavity where photons are localized about point defects
- Likewise for phonons

How does the electric field (displacement field) look like?

Re $\psi_{n,k}$



Wavepacket dynamics

I can study the dynamics of a wavepacket by decomposing it in terms of Bloch waves

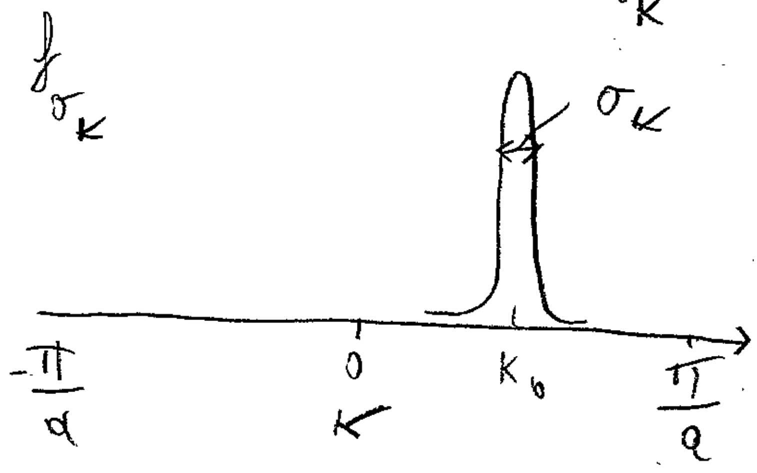
$$\Psi(x,t) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk\, f(k)\, \psi_{n,k}(x)\, e^{-i\omega_{n,k} t}$$

↗ wavepacket

$$= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} f(k)\, \phi_{n,k}(x)\, e^{i(kx - \omega_{n,k} t)}$$

Consider $f(k)$ localized about k_0, e.g. gaussian

$$f(k) = f_{\sigma_k}(k) = \frac{1}{\sqrt{2\pi\sigma_k^2}}\, e^{-\frac{(k-k_0)^2}{2\sigma_k^2}}$$



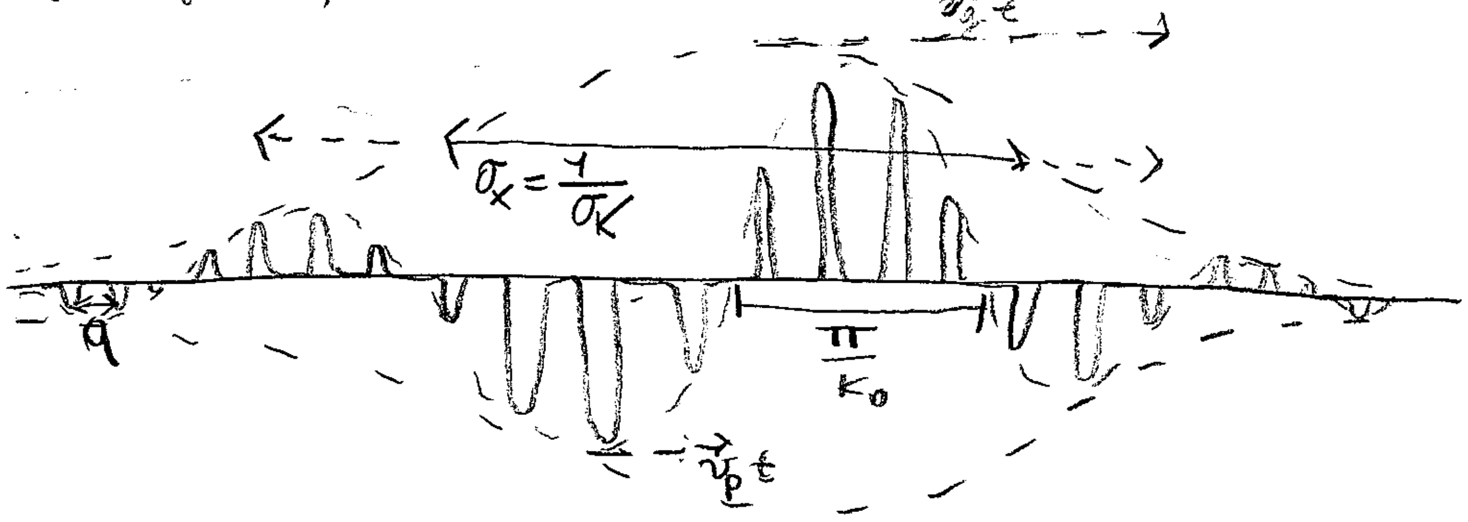
If $\sigma_k \ll |k_0|, \frac{\pi}{a}$ I can neglect the dependence of $\phi_{n,k}(x)$ on k and sent the integration limits to infinity

$$\Psi(x,t) \simeq \phi_{n,k_0}(x) \boxed{\int_{-\infty}^{\infty} dk\, f_{\sigma_k}(k)\, e^{i(kx - \omega_{n,k} t)}}$$

↙

Same expression for wave-packet, e.g. light, in a dispersive medium. Here, it is modulated by $\phi_{n,k_0}(x)$, see sketch!

$\mathrm{Re}\,\Psi(x,0)$ (Fourier transform of a Gaussian is a Gaussian)



Time evolution:

(can be calculated by expanding $\omega_{n,k}$ about k_0,

$$\omega_{n,k} \approx \omega_{n,k_0} + \left.\frac{\partial \omega_{n,k}}{\partial k}\right|_{k_0} (k - k_0) + \frac{1}{2} \frac{\partial^2 \omega_{n,k}}{\partial k^2} (k - k_0)^2)$$

- Peak of gaussian envelope moves with group velocity $v_g = \left.\frac{\partial \omega_{n,k}}{\partial k}\right|_{k=k_0}$ (in 2D or 3D $\vec{v}_g = \left.\vec{\nabla} \omega_{n,k}\right|_{k=k_0}$)

- The peak of the sinusoidal envelope moves with phase velocity $v_p = \omega_{n,k_0} / k_0$

- Dispersion: spread of wave packet increases with time for short times $t \ll \left(\sigma_k^2 \frac{\partial^2 \omega}{\partial k^2}\right)^{-1}$

$$\sigma_x(t) \approx \frac{1}{\sigma_K} + c\,t^2 \qquad c = \sigma_K^3 \left(\frac{\partial^2 \omega}{\partial k^2}\right)^2$$