Notes on cavity (hybrid) optomechanics

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(Dated: July 6, 2017)

We describe steady state solutions for a cavity with a movable mirror in the presence of an ensemble of two-level systems (atoms, molecules, ions etc). The derivation uses the quantum Langevin equations formalism and proceeds with solving a set of coupled linearized equations in the Fourier domain. The (hybrid) cooling rates are derived. More details can be found in Refs. [1, 2].

CONTENTS

INTRODUCTION

We consider a vibrating end-mirror of an optical cavity subject to radiation pressure forces stemming from the intracavity photons. We first describe Langevin equations (Heisenberg equations of motion supplemented with quantum noise terms) for the cavity mode and derive cavity output and noise spectra. We then proceed to include the mirror motion and the mirror backaction onto the cavity field. We denote this by a standard cavity optomechanics regime, and derive the mirror response where the thermal noise and damping rate are modified with terms coming from the cavity properties. Finally, we include an ensemble of atoms in the cavity which modify the cavity mode properties (cavity output, cavity mode noise spectrum). Following the standard optomechanics formalism, we show that the hybrid setup can improve on the standard one, by allowing one to efficiently suppress heating Stokes sidebands or enhance cooling Antistokes sidebands.

BARE CAVITY RESPONSE

Let us review some fundamental results for a single mode (annihilation operator a, frequency ω_c and decay rate κ) driven optical cavity (at laser frequency ω_l). We start with the Hamiltonian

$$H = \hbar\omega_c a^\dagger a - i\hbar\mathcal{E}(a^\dagger e^{-i\omega_l t} - a e^{i\omega_l t}) \tag{1}$$

where the driving amplitude is connected to the cavity laser input power as

$$\mathcal{E} = \sqrt{\frac{2P\kappa}{\hbar\omega_l}}$$

To describe the dynamics we chose to write equations of motion supplemented by quantum (quantum Langevin equations)

$$\dot{a} = -(\kappa + i\Delta_f)a + \mathcal{E} + \sqrt{2\kappa}a_{in} \tag{2}$$

where we have already moved into a rotating frame such that the detuning is $\Delta_f = \omega_c - \omega_l$ and the noise has zero average and is assumed to be white noise such that the only non-vanishing correlations is $\left\langle a_{in}(t)a_{in}^\dagger(t')\right\rangle = \delta(t-t')$.

Making use of the input-output relation

$$a_{in} + a_{out} = \sqrt{2\kappa}a, \tag{3}$$

we can estimate the average of the output field amplitude by setting steady state conditions (zero derivative)

$$\langle a_{out}\rangle = \sqrt{2\kappa}\,\langle a\rangle = \frac{\sqrt{2\kappa}\mathcal{E}}{\kappa + i\Delta_f}. \tag{4}$$

We can see that first that the output field reproduces the intracavity field and its intensity follows a Lorentzian exhibiting a resonance at the cavity frequency ω_c (assuming one scans the laser frequency from one side to the other of the cavity resonance).
We now want to describe the cavity field quantum noise. To this end we subtract the classical average in steady state and describe the properties of the zero-mean fluctuations $\delta a(t)$:

$$a(t) = \delta a(t) + \langle a\rangle\,, \tag{5}$$

following the corresponding equation

$$\dot{\delta a} = -(\kappa + i\Delta_f)\delta a + \sqrt{2\kappa}a_{in}. \tag{6}$$

We will in the following make extensive use of the Fourier transform which we define as:

$$\delta a(\omega) = \mathcal{F}[\delta a(t)] = \frac{1}{\sqrt{2\pi}}\int dt\delta a(t)e^{-i\omega t}. \tag{7}$$

Notice an important property when performing Hermitian conjugation: $\delta a^\dagger(\omega) = \mathcal{F}[\delta a^\dagger(t)] = [\delta a(-\omega)]^\dagger$ (The hermitian conjugate of the Fourier transform is different than the Fourier transform of the hermitian conjugate).

The Langevin equation in the Fourier domain is now:

$$-i\omega\delta a(\omega) = -(\kappa + i\Delta_f)\delta a(\omega) + \sqrt{2\kappa}a_{in}(\omega), \tag{8}$$

and the solution can be simply expressed

$$\boxed{\delta a(\omega) = \frac{\sqrt{2\kappa}a_{in}(\omega)}{\kappa + i(\Delta_f - \omega)} = \sqrt{2\kappa}\epsilon_f(\omega)a_{in}} \tag{9}$$

in terms of the bare cavity susceptibility:

$$\boxed{\epsilon_f(\omega) = \frac{1}{\kappa + i(\Delta_f - \omega)}}. \tag{10}$$

Notice that the Fourier component of the hermitian conjugate contains an important minus sign:

$$\boxed{\delta a^\dagger(\omega) = \frac{\sqrt{2\kappa}a_{in}^\dagger(\omega)}{\kappa - i(\Delta_f + \omega)} = \sqrt{2\kappa}\epsilon_f^*(-\omega)a_{in}^\dagger} \tag{11}$$

Exercise 1: Given the input field correlations in the Fourier domain (input field noise spectrum) $\left\langle \delta a_{in}(\omega)\delta a_{in}^\dagger(\omega')\right\rangle = \delta(\omega+\omega')$, compute the output field noise spectrum. Step 1: compute the intracavity spectrum: $\left\langle \delta a(\omega)\delta a^\dagger(\omega')\right\rangle = |\epsilon_f(\omega)|^2\delta(\omega+\omega')$.
______________________________ End exercise 1.

OPTOMECHANICS

We now allow the end-mirror to vibrate, modulating the cavity length and therefore coupling to the light field as follows:

$$H = \hbar\omega_c a^\dagger a + \frac{1}{2}\hbar\omega_m(p^2+q^2) - \hbar G_0 a^\dagger a q + i\hbar\mathcal{E}(a^\dagger e^{-i\omega_l t} - a e^{i\omega_l t}), \tag{12}$$

where the mirror is described by the conjugate quadrature operators q, p (normalized such that they are dimensionless) and the optomechanical coupling scales as:

$$G_0 = \frac{\omega_c}{L}\sqrt{\frac{\hbar}{m\omega_m}}, \tag{13}$$

with the cavity length L, and the mirror mass m and vibration frequency ω_m. As before we proceed with writing Langevin equations

$$\begin{aligned}
\dot{a} &= -(\kappa + i\Delta_f)a + iG_0 a q + \mathcal{E} + a_{in},\\
\dot{p} &= -i\omega_m q - \gamma_m p + G_0 a^\dagger a + p_{in},\\
\dot{q} &= \omega_m p.
\end{aligned}$$

The noise term affecting the mechanical resonator is non-Markovian and is generally written as:

$$\langle p_{in}(t)p_{in}(t')\rangle = \frac{\gamma_m}{\omega_m}\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\omega\left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)+1\right],$$

where k_B is the Boltzmann constant and T is the environmental temperature. For room-temperature conditions and mechanical frequency of the order of MHz-GHz, $k_B T \gg \hbar\omega$ and the $\coth[\hbar\omega/(2k_B T)] \simeq 2k_B T/(\hbar\omega_m)$. Rewriting the correlations in the Fourier domain and under large T conditions:

$$\langle p_{in}(\omega)p_{in}(\omega')\rangle = \frac{\gamma_m}{\omega_m}\omega\left\{\coth\left[\frac{\hbar\omega}{2k_B T}\right]+1\right\} \longrightarrow \gamma_m\left\{\frac{2k_B T}{\hbar\omega_m}+1\right\}.$$

The equations above are non-linear. We proceed to simplify them by performing a linearization around expectation values in steady state

$$\begin{aligned}
a &= \langle a\rangle + \delta a,\\
p &= \langle p\rangle + \delta p,\\
q &= \langle q\rangle + \delta q.
\end{aligned}$$

Exercise 2: Find the expectation values $\langle a\rangle$, $\langle q\rangle$ and $\langle p\rangle$ in steady state. Hint: set derivatives to zero.
____ End exercise 2.

We then find the following set of linear Langevin equations for fluctuations (eliminating small contributions from terms containing products of fluctuations):

$$\begin{aligned}
\dot{\delta a} &= -(\kappa + i\Delta_f)\delta a + iG_0\langle q\rangle + iG_0\langle a\rangle q + \sqrt{2\kappa}a_{in},\\
\dot{\delta p} &= -i\omega_m\delta q - \gamma_m\delta p + G_0\langle a\rangle(\delta a^\dagger + \delta a) + p_{in},\\
\dot{\delta q} &= \omega_m\delta p.
\end{aligned}$$

Let's make the notations $G = G_0\langle a\rangle$ (collectively enhanced optomechanical coupling) and $\bar{\Delta}_f = \Delta_f - G_0\langle q\rangle$ we arrive at the Fourier domain equations:

$$\begin{aligned}
-i\omega\delta q(\omega) &= \omega_m\delta p(\omega),\\
-i\omega\delta p(\omega) &= -i\omega_m\delta q(\omega) - \gamma_m\delta p(\omega) + G\left[\delta a^\dagger(\omega) + \delta a(\omega)\right] + p_{in}(\omega),\\
-i\omega\delta a(\omega) &= -(\kappa + i\bar{\Delta}_f)\delta a(\omega) + iG\delta q(\omega) + \sqrt{2\kappa}a_{in}(\omega).
\end{aligned}$$

We focus on the equation for the position quadrature which we can recast as:

$$[\epsilon_m(\omega)]^{-1}\,\delta q(\omega) = G\left[\delta a^\dagger(\omega) + \delta a(\omega)\right] + p_{in}(\omega),$$

by making use of the defined mechanical susceptibility (and notice that in the last equation I flipped the sign of ω as $[\delta q(\omega)]^\dagger = \delta q(-\omega)$).

$$\epsilon_m(\omega) = \frac{\omega_m}{(\omega_m^2 - \omega^2) - i\gamma_m\omega}.$$

Exercise 3: Bare harmonic oscillator in thermal equilibrium: Find the position quadrature variance in for the bare harmonic oscialltor in thermal equilibrium. Setting $G = 0$ we simply have

$$\delta q(\omega) = \epsilon_m(\omega) p_{in}(\omega),$$

We can compute

$$\begin{aligned}
\langle(\delta q)^2\rangle(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega d\omega' \langle\delta q(\omega)\delta q(\omega')\rangle e^{-i(\omega+\omega')t} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega d\omega' \epsilon_m(\omega)\epsilon_m(\omega')\langle p_{in}(\omega)p_{in}(\omega')\rangle e^{-i(\omega+\omega')t} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \epsilon_m(\omega)\epsilon_m(-\omega) S_{th}(\omega).
\end{aligned}$$

Notice that $\epsilon_m(-\omega) = \epsilon_m^*(\omega)$ and let us consider the high-temperature case where

$$S_{th}(\omega) = \frac{\gamma_m}{\omega_m}\omega\left\{\coth\left[\frac{\hbar\omega}{2K_BT}\right] + 1\right\} \longrightarrow \gamma_m\left\{\frac{2K_BT}{\hbar\omega_m} + 1\right\}.$$

From the normalization $\int_{-\infty}^{\infty} d\omega|\epsilon_m(\omega)|^2 = \pi/\gamma_m$, we have the result for the high temperature limit:

$$\langle(\delta q)^2\rangle(t) = \frac{K_BT}{\hbar\omega_m} + 1/2 = n_{th} + 1/2..$$

________________________________ End exercise 3.

The extra driving terms are proportional to field fluctuations given by: Solving

$$\begin{aligned}
[\epsilon_f(\omega)]^{-1}\,\delta a(\omega) &= iG\delta q(\omega) + \sqrt{2\kappa}a_{in}(\omega), \\
\left[\epsilon_f^*(-\omega)\right]^{-1}\,\delta a^\dagger(\omega) &= -iG\delta q(\omega) + \sqrt{2\kappa}a_{in}^\dagger(\omega).
\end{aligned}$$

Let's proceed with solving for this:

$$\boxed{\left\{[\epsilon_m(\omega)]^{-1} - iG^2\left[\epsilon_f(\omega) - \epsilon_f^*(-\omega)\right]\right\}\delta q(\omega) = G\sqrt{2\kappa}\left[\epsilon_f(\omega)a_{in}(\omega) + \epsilon_f^*(-\omega)a_{in}^\dagger(\omega)\right] + p_{in}(\omega).}$$

On the left side there is a modified susceptibility which we can write as

$$\boxed{[\bar{\epsilon}_m(\omega)]^{-1} = [\epsilon_m(\omega)]^{-1} - iG^2\left[\epsilon_f(\omega) - \epsilon_f^*(-\omega)\right].}$$

while on the right side there is a modified noise term:

$$\boxed{\bar{p}_{in}(\omega) = G\sqrt{2\kappa}\left[\epsilon_f(\omega)a_{in}(\omega) + \epsilon_f^*(-\omega)a_{in}^\dagger(\omega)\right] + p_{in}(\omega).}$$

In simplified notation we then have:

$$\boxed{\delta q(\omega) = \bar{\epsilon}_m(\omega)\bar{p}_{in}(\omega).}$$

Exercise 4: Cooling rate: Find the cavity-mediated cooling rates. Hint: Analyze the modified mechanical susceptibility to find a damping rate γ_m + contributions proportional to G^2

Expanding the imaginary part of the effective mechanical susceptibility we obtain:

$$\boxed{\gamma_{eff}(\omega) = \gamma_m + \frac{G^2\omega_m}{\kappa\omega}\left[\frac{1}{\kappa^2 + (\Delta_f - \omega)^2} - \frac{1}{\kappa^2 + (\Delta_f + \omega)^2}\right]}$$

As the mechanical susceptibility is pretty much zero everywhere except around $\omega = \omega_m$ let's see what the optical cooling rate looks like evaluated there:

$$\boxed{\gamma_{eff}(\omega_m) = \gamma_m + \frac{G^2}{\kappa}\left[\frac{1}{\kappa^2 + (\Delta_f - \omega_m)^2} - \frac{1}{\kappa^2 + (\Delta_f + \omega_m)^2}\right]}.$$

Rewriting to make it more clear:

$$\boxed{\gamma_{eff}(\omega_m) = \gamma_m + \frac{G^2}{\kappa}\frac{4\omega_m\Delta_f}{[\kappa^2 + (\Delta_f - \omega_m)^2]\,[\kappa^2 + (\Delta_f + \omega_m)^2]}},$$

we clearly see that the sign of Δ_f is crucial for the optical contribution to the damping (plus or minus corresponds to cooling or heating). This is the famous sideband cooling result. For $\kappa \gg \omega_m$ we can estimate

$$\boxed{\gamma_{eff} \simeq \gamma_m + \frac{G^2}{\kappa}\left(\frac{2\omega_m}{\kappa}\right)^2},$$

$\gamma_{eff} \simeq \gamma_m + 4G^2/\kappa$. This is known as unresolved sideband cooling and it corresponds to the bad cavity regime. In the opposite limit we have

$$\boxed{\gamma_{eff} \simeq \gamma_m + \frac{G^2}{\kappa}\frac{4\omega_m^2}{\kappa^2 + 4\omega_m^2}},$$

______________________________ End exercise 4.

Exercise 5: Intracavity photons Starting with

$$[\epsilon_f(\omega)]^{-1}\,\delta a(\omega) = iG\delta q(\omega) + \sqrt{2\kappa}a_{in}(\omega),$$
$$\left[\epsilon_f^*(-\omega)\right]^{-1}\delta a^\dagger(\omega) = -iG\delta q(\omega) + \sqrt{2\kappa}a_{in}^\dagger(\omega),$$
$$\delta q(\omega) = \bar{\epsilon}_m(\omega)\bar{p}_{in}(\omega).$$

we can compute

$$\left[\epsilon_f(\omega')\epsilon_f^*(-\omega)\right]^{-1}\langle\delta a^\dagger(\omega')\delta a(\omega)\rangle = ...$$

______________________________ End exercise 5.

ATOMS IN CAVITIES

We now want to provide an equivalent description for the interaction of a single cavity mode with intracavity atoms. We will write the Hamiltonian

$$H = \hbar\omega_c a^\dagger a + \sum_{j=1}^{N} \hbar\omega_a \sigma_z^j + \sum_{j=1}^{N} \hbar g_j (\sigma_j a^\dagger + \sigma_j^\dagger a) + i\hbar\mathcal{E}(a^\dagger e^{-i\omega_l t} - a e^{i\omega_l t}), \tag{14}$$

where the new terms represent the free Hamiltonian for N identical two-level systems of energy $\hbar\omega_a$ and the $\sigma_z, \sigma and \sigma^\dagger$ are Pauli matrices. The full dynamics is complicated as two-level system are intrinsically non-linear systems. We however can linearize assuming that on average the collection is weakly excited and replace the collective excitations with a bosonic creation operator:

$$c = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_j.$$

In such a case the total excitation is simply a bosonic displacement from the new ground state at energy $-N/2\hbar\omega_a$. We can do as before writing Langevin equations:

$$\dot{\delta a} = -(\gamma + i\Delta_c)\delta a - iG_a \delta c + \sqrt{2\kappa} a_{in},$$
$$\dot{\delta c} = -(\gamma + i\Delta_a)\delta c - iG_a \delta a + \sqrt{2\gamma} c_{in},$$

where the atomic input noise has the usual delta-correlations in the time domain $\left\langle c_{in}(t) c_{in}^\dagger(t') \right\rangle = \delta(t-t')$. In the Fourier domain we have

$$-i\omega\delta a(\omega) = -(\kappa + i\Delta_f)\delta a(\omega) - iG_a \delta c(\omega) + \sqrt{2\kappa} a_{in}(\omega),$$
$$-i\omega\delta c(\omega) = -(\gamma + i\Delta_a)\delta c(\omega) - iG_a \delta a(\omega) + \sqrt{2\gamma} c_{in}(\omega),$$

which can be cast in a very simple form introducing a new susceptibility, that of the atom:

$$[\epsilon_f(\omega)]^{-1} \delta a = -iG_a \delta c + \sqrt{2\kappa} a_{in},$$
$$[\epsilon_a(\omega)]^{-1} \delta c = -iG_a \delta a + \sqrt{2\gamma} c_{in}.$$

The atom susceptibility:

$$\boxed{\epsilon_a(\omega) = \frac{1}{\gamma + i(\omega_a - \omega)}.}$$

However, we are not interested in atomic degrees of freedom but rather looking for an effective description of the cavity dressed by the atoms. Therefore we solve for:

$$[\epsilon_f(\omega)]^{-1} \delta a = -iG_a(-iG_a \epsilon_a(\omega)\delta a + \epsilon_a(\omega)\sqrt{2\gamma} c_{in}) + \sqrt{2\kappa} a_{in},$$

with the result

$$\boxed{\left\{ [\epsilon_f(\omega)]^{-1} + G_a^2 \epsilon_a(\omega) \right\} \delta a = -iG_a \epsilon_a(\omega) \sqrt{2\gamma} c_{in} + \sqrt{2\kappa} a_{in}},$$

which, as before, can be written very simply:

$$\boxed{\delta a = \sqrt{2\kappa} \bar{\epsilon}_f(\omega) \bar{a}_{in}},$$

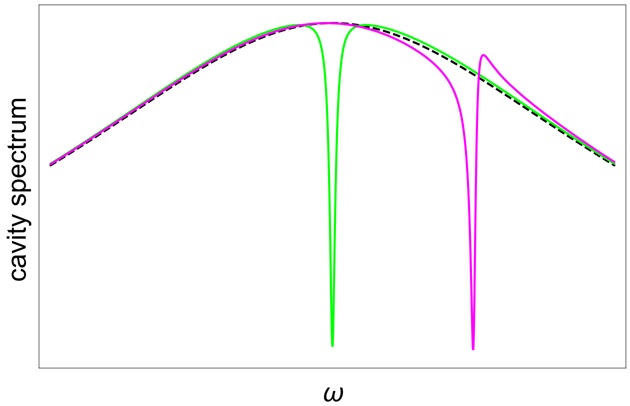

FIG. 1. Cavity spectrum without atoms (black, dashed line), with atoms on resonance with the cavity (green line) and with atoms off-resonant with the cavity (magenta line). Notice that the dip in spectrum is positioned around the atomic resonance and has a linewidth roughly proportional to the atomic linewidth.

with the atom-dressed field susceptibility

$$\boxed{[\bar{\epsilon}_f(\omega)]^{-1} = [\epsilon_f(\omega)]^{-1} + G_a^2 \epsilon_a(\omega)},$$

and the total field-atom input noise:

$$\boxed{\bar{a}_{in} = -iG_a \epsilon_a(\omega)\sqrt{2\gamma}c_{in} + \sqrt{2\kappa}a_{in}},$$

HYBRID (ATOM) OPTOMECHANICS

Let us extend the model to atoms, cavity mode and mirror motion:

$$H = \hbar\omega_c a^\dagger a + \hbar\omega_a c^\dagger c + \frac{1}{2}\hbar\omega_m(p^2+q^2) - \hbar G_0 a^\dagger a q + \hbar G_a(a^\dagger c + c^\dagger a) + i\hbar\mathcal{E}(a^\dagger e^{-i\omega_l t} - a e^{i\omega_l t}). \tag{15}$$

Performing all the tricks introduced in the previous sections we end up with the following set of equations in the Fourier domain:

$$\begin{aligned}
-i\omega\delta a(\omega) &= -(\kappa + i\bar{\Delta}_f)\delta a(\omega) - iG_a\delta c(\omega) + iG\delta q(\omega) + \sqrt{2\kappa}a_{in}(\omega),\\
-i\omega\delta c(\omega) &= -(\gamma + i\Delta_a)\delta c(\omega) - iG_a\delta a(\omega) + \sqrt{2\gamma}c_{in}(\omega),\\
-i\omega\delta p(\omega) &= -i\omega_m\delta q(\omega) - \gamma_m\delta p(\omega) + G\left[\delta a^\dagger(\omega) + \delta a(\omega)\right] + p_{in}(\omega),\\
-i\omega\delta q(\omega) &= \omega_m\delta p(\omega).
\end{aligned}$$

As before we then redefine an effective mechanical susceptibility

$$[\bar{\bar{\epsilon}}_m(\omega)]^{-1} = [\epsilon_m(\omega)]^{-1} - iG^2\left[\bar{\epsilon}_f(\omega) - \bar{\epsilon}_f^*(-\omega)\right],$$

which now contains the modification brought on by the atoms as well

$$\boxed{[\bar{\epsilon}_f(\omega)]^{-1} = [\epsilon_f(\omega)]^{-1} + G_a^2\epsilon_a(\omega)},$$

Moreover, we also have the modified input noises:

$$\bar{\bar{p}}_{in}(\omega) = G\sqrt{2\kappa}\left[\bar{\epsilon}_f(\omega)\bar{a}_{in}(\omega) + \bar{\epsilon}_f^*(-\omega)\bar{a}_{in}^\dagger(\omega)\right] + p_{in}(\omega).$$

As before, we look at the imaginary part of the effective mechanical susceptibility to find the cooling rate introduced by the atom-field system. If $\gamma \ll \omega_m$ and we set $\delta = -\omega_m$ (corresponding to the atom fitting the Stokes sideband $\omega_a = \omega_l - \omega_m$), we can approximate $\epsilon_a(\omega_m) \simeq 1/(2\omega_m)$, $\epsilon_a(-\omega_m) \simeq 1/\gamma$, $\bar{\epsilon_f}(\omega_m) \simeq 1/\kappa$ and $\bar{\epsilon}_f(-\omega_m) \simeq 1/(\kappa + C)$ where the cooperativity is defined as $C = g^2N/(\kappa\gamma)$.
In consequence:

$$\boxed{\gamma_{eff}(\omega_m) = \gamma_m + \frac{G^2}{\kappa}\frac{C}{1+C}},$$

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[2] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer *Ground-state cooling of a micromechanical oscillator: Comparing cold damping and cavity-assisted cooling schemes* Phys. Rev. A **77**, 033804 (2008).